

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2019: 45(9), p. 518–523.

4481. *Proposed by Warut Suksompong.*

Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^2 + y^2) = f(x + y)f(x - y) + 2f(y)y$$

for all $x, y \in \mathbb{R}$.

We received 16 submissions, of which 7 were correct and complete. We present the solution by Michel Bataille.

There are three functions that satisfy the equation:

- $f_0(x) = 0$ for all $x \in \mathbb{R}$,
- $f_1(x) = x$,
- $f_2(x) = 0$ for $x \neq 0$ and $f_2(0) = 1$.

The functions f_0, f_1 are obvious solutions. As for the function f_2 , consider first the case when $x^2 + y^2 = 0$. Then $x = y = 0$ and

$$f_2(x^2 + y^2) = f_2(0) = 1 = f_2(0)f_2(0) + 2f_2(0) \cdot 0 = f_2(x + y)f_2(x - y) + 2f_2(y)y.$$

If $x^2 + y^2 \neq 0$, then $x + y$ and $x - y$ cannot equal 0 both, hence $f_2(x + y)f_2(x - y) = 0$ and $2f_2(y)y = 0$ (since $f_2(y) = 0$ if $y \neq 0$) so that the required functional equation still holds and f_2 is also a solution.

Conversely, let f be an arbitrary solution. For convenience, we let $E(x, y)$ denote the equality $f(x^2 + y^2) = f(x + y)f(x - y) + 2f(y)y$. From $E(0, 0)$ we deduce that $f(0) = (f(0))^2$, hence $f(0) = 0$ or $f(0) = 1$.

First, suppose that $f(0) = 0$. From $E(x, x)$ and $E(x, -x)$ we obtain

$$2xf(x) = -2xf(-x) = f(2x^2),$$

so that $f(-x) = -f(x)$ if $x \neq 0$ and f is an odd function. With the help of this result and comparing $E(x, y)$ and $E(y, x)$, we get that

$$f(x + y)f(x - y) = xf(x) - yf(y)$$

for all x, y . Taking $y = 0$, this yields $(f(x))^2 = xf(x)$ and so $f(x) = 0$ or $f(x) = x$ for all x . If $f \neq f_0$, we have $f(a) = a \neq 0$ for some a . Now, we consider $x \neq 0$ and assume that $f(x) = 0$. Then, $f(x + a)f(x - a) = xf(x) - a^2 = -a^2 \neq 0$

and we must have $f(x+a) = x+a$ and $f(x-a) = x-a$. But it follows that $(x+a)(x-a) = -a^2$, contradicting $x \neq 0$. In consequence $f(x) = x$ and we can conclude that $f = f_0$ or $f = f_1$ in the case when $f(0) = 0$.

Second, suppose $f(0) = 1$. Again $E(x, x)$ and $E(x, -x)$ give $xf(x) = -xf(-x)$ so that $f(-x) = -f(x)$ whenever $x \neq 0$. Comparing $E(x, y)$ and $E(y, x)$ now gives $f(x+y)f(x-y) = xf(x) - yf(y)$ when $x \neq y$. To conclude, consider $x \neq 0$. Then $y = -\frac{x}{2} \neq \frac{x}{2}$, hence

$$f\left(\frac{x}{2} + \frac{x}{2}\right)f\left(\frac{x}{2} - \frac{x}{2}\right) = \frac{x}{2}f\left(\frac{x}{2}\right) + \frac{x}{2}f\left(-\frac{x}{2}\right),$$

that is, $f(x) = 0$. Thus, $f = f_2$ when $f(0) = 1$ and the proof is complete.

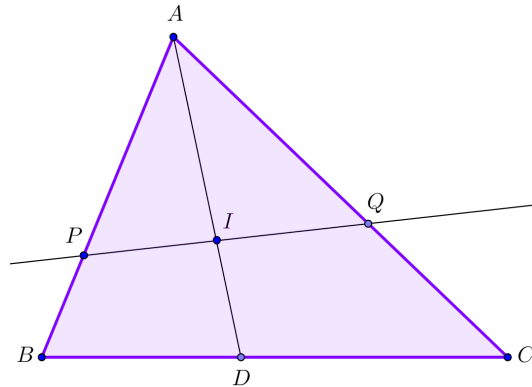
Editor's comments. With respect to incorrect solutions, there were two common errors: to note that if $f(0) = 0$ then for $x \neq 0$ it must be true that either $f(x) = x$ or $f(x) = 0$, but to omit the explanation of why only one of these statements must hold for all $x \neq 0$, and to miss the solution which has $f(0) = 1$.

4482. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let ABC be a triangle with incenter I . The line AI intersects BC at D . A line l passes through I and intersects the sides AB and AC at P and Q , respectively. Show that

$$AC \cdot \frac{[BDIP]}{[API]} + AB \cdot \frac{[CDIQ]}{[AQI]} = 2 \cdot BC + \frac{BC^2}{AB + AC},$$

where square brackets denote area.



All 18 of the submissions we received provided complete solutions; we feature the solution by Marie-Nicole Gras.

We put

$$a = BC, b = CA, c = AB, u = AP, v = AQ,$$

and use the label A to also denote the angles $\angle BAC = \angle PAQ$. The desired result is a consequence of the following known facts:

$$AD = \frac{2bc}{b+c} \cos \frac{A}{2}, \quad AI = \frac{2bc}{a+b+c} \cos \frac{A}{2}, \quad \text{so that } \frac{AD}{AI} = \frac{a+b+c}{b+c}.$$

Moreover, since PQ passes through I , AI is the internal bisector of the angle at A in $\triangle PAQ$, so that

$$AI = \frac{2uv}{u+v} \cos \frac{A}{2} = \frac{2bc}{a+b+c} \cos \frac{A}{2}, \quad \text{which implies } \frac{u+v}{uv} = \frac{a+b+c}{bc}.$$

Putting these facts together, we have

$$\begin{aligned} b \frac{[BDIP]}{[API]} + c \frac{[CDIQ]}{[AQI]} &= b \frac{[ABD] - [API]}{[API]} + c \frac{[ACD] - [AQI]}{[AQI]} \\ &= b \frac{cAD \sin \frac{A}{2} - uAI \sin \frac{A}{2}}{uAI \sin \frac{A}{2}} + c \frac{bAD \sin \frac{A}{2} - vAI \sin \frac{A}{2}}{vAI \sin \frac{A}{2}} \\ &= \frac{bc}{u} \frac{AD}{AI} + \frac{bc}{v} \frac{AD}{AI} - b - c \\ &= bc \frac{AD}{AI} \left(\frac{u+v}{uv} \right) - b - c \\ &= bc \frac{a+b+c}{b+c} \frac{a+b+c}{bc} - b - c \\ &= \frac{(a+b+c)^2}{b+c} - (b+c) \\ &= \frac{(a+b+c)^2 - (b+c)^2}{b+c} \\ &= \frac{a(a+2b+2c)}{b+c} = 2a + \frac{a^2}{b+c} = 2BC + \frac{BC^2}{AB+AC}. \end{aligned}$$

4483. *Proposed by Paul Bracken.*

For non-negative integers m and n , evaluate the following sum in closed form

$$\sum_{j=0}^m j^2 \binom{j+n}{j}.$$

We received 15 solutions, all correct. We present the solution by Marie-Nicole Gras.

We will use the hockey-stick identity

$$\sum_{k=0}^m \binom{n+k}{n} = \binom{m+n+1}{n+1}.$$

For all $m, n \geq 0$ let $S_{m,n} = \sum_{j=0}^m j^2 \binom{j+n}{j}$. We have

$$\begin{aligned} S_{m,n} &= \sum_{j=0}^m [j + j(j-1)] \binom{n+j}{n} \\ &= \sum_{j=1}^m j \binom{n+j}{n} + \sum_{j=2}^m j(j-1) \binom{n+j}{n} \\ &= \sum_{j=1}^m (n+1) \binom{n+j}{n+1} + \sum_{j=2}^m (n+1)(n+2) \binom{n+j}{n+2} \\ &= (n+1) \sum_{k=0}^{m-1} \binom{n+1+k}{n+1} + (n+1)(n+2) \sum_{k=0}^{m-2} \binom{n+2+k}{n+2}. \end{aligned}$$

Apply the hockey-stick identity to each sum to get

$$S_{m,n} = (n+1) \binom{m+n+1}{n+2} + (n+1)(n+2) \binom{m+n+1}{n+3}.$$

4484. Proposed by Leonard Giugiuc and Michael Rozenberg.

Let $a, b, c \in [0, 2]$ such that $a + b + c = 3$. Prove that

$$4(ab + bc + ac) \leq 12 - ((a-b)(b-c)(c-a))^2$$

and find when the equality holds.

We received nine submissions, eight of which are correct and the other is incomplete. We present the solution by Digby Smith, modified slightly by the editor.

Without loss of generality we may assume that $a \geq b \geq c$.

Let $x = a - b$, $y = b - c$, and $z = a - c$. Then $x, y, z \geq 0$ and $x + y = z \leq 2$. Hence, $4xy \leq (x + y)^2 \leq 4$ so $xy \leq 1$, with equality if and only if $x = y = 1$.

Next, note that

$$\begin{aligned} xy - (x + y)^2 &= xy - z(x + y) = xy - yz - zx \\ &= (a-b)(b-c) + (b-c)(c-a) + (c-a)(a-b) \\ &= (ab + bc + ca) - (a^2 + b^2 + c^2) \\ &= (ab + bc + ca) - ((a+b+c)^2 - 2(ab + bc + ca)) \\ &= 3((ab + bc + ca) - 3). \end{aligned} \tag{1}$$

By (1), the given inequality is equivalent in succession to

$$\begin{aligned} 12((ab + bc + ca) - 3) + 3(a - b)^2(b - c)^2(c - a)^2 &\leq 0, \\ 4(xy - (x + y)^2) + 3x^2y^2(x + y)^2 &\leq 0, \\ (x + y)^2(4 - 3x^2y^2) - 4xy &\geq 0. \end{aligned} \quad (2)$$

Finally, since $xy \leq 1$, we have

$$\begin{aligned} (x + y)^2(4 - 3x^2y^2) - 4xy &\geq 4xy(4 - 3x^2y^2) - 4xy \\ &= 4xy(3 - 3x^2y^2) \\ &= 12xy(1 - xy)(1 + xy) \geq 0, \end{aligned}$$

so (2) holds and the proof of the inequality is complete. Clearly, equality holds if and only if $x = y = 0$ or $x = y = 1$ which implies that $(a, b, c) = (1, 1, 1)$ or any permutation of $(2, 1, 0)$.

4485. *Proposed by Jonathan Parker and Eugen J. Ionascu.*

For every square matrix with real entries $A = [a_{i,j}]_{i=1..n, j=1,2..n}$, we define the value

$$GM(A) = \max_{\pi \in S_n} \{\min\{a_{1\pi(1)}, a_{2\pi(2)}, \dots, a_{n\pi(n)}\}\}$$

where S_n is the set of all permutations of the set $[n] := \{1, 2, 3, \dots, n\}$.

Given the 6×6 matrix

$$A := \begin{bmatrix} 20 & 9 & 7 & 26 & 27 & 13 \\ 19 & 18 & 17 & 6 & 12 & 25 \\ 22 & 24 & 21 & 11 & 20 & 11 \\ 20 & 8 & 9 & 23 & 5 & 14 \\ 22 & 17 & 4 & 10 & 36 & 33 \\ 21 & 16 & 23 & 35 & 15 & 34 \end{bmatrix}$$

find the value $GM(A)$.

There were 11 correct solutions, most straightforward.

Let $\pi = (15)(263)$. Then

$$\begin{aligned} \min\{a_{1,\pi(1)}, a_{2,\pi(2)}, \dots, a_{6,\pi(6)}\} &= \min\{a_{15}, a_{26}, a_{32}, a_{44}, a_{51}, a_{63}\} \\ &= \min\{27, 25, 24, 23, 22, 23\} \\ &= 22. \end{aligned}$$

Therefore $GM(A) \geq 22$.

On the other hand, since for each permutation π , the set being minimized contains a number $a_{\pi^{-1}(1),1}$ from the first column, the minimum cannot exceed 22. Thus $GM(A) \leq 22$. Therefore $GM(A) = 22$.

4486. *Proposed by Marian Cucoaneş and Marius Drăgan.*

Let $a, b > 0$, $c > 1$ such that $a^2 \geq b^2c$. Compute

$$\lim_{n \rightarrow \infty} (a - b\sqrt{c})(a - b\sqrt[3]{c}) \cdots (a - b\sqrt[n]{c}).$$

We received 12 solutions, one of which was incomplete and one of which was incorrect. We present the solution by Michel Bataille.

Let $P_n = \prod_{k=2}^n (a - b\sqrt[k]{c})$. From the hypotheses, we have $a \geq b\sqrt{c}$. We show that if $a = b\sqrt{c}$, then $\lim_{n \rightarrow \infty} P_n = 0$ while if $a > b\sqrt{c}$, then $\lim_{n \rightarrow \infty} P_n = \infty$ if $a > b + 1$ and $\lim_{n \rightarrow \infty} P_n = 0$ if $a \leq b + 1$.

If $a = b\sqrt{c}$, then $P_n = 0$ for all $n \geq 2$ and so $\lim_{n \rightarrow \infty} P_n = 0$.

From now on, we suppose that $a > b\sqrt{c}$. Note that $a > b$ and that $a - b\sqrt[k]{c} > a - b\sqrt{c} > 0$ for all $k \geq 2$ (so that $P_n > 0$ for all $n \geq 2$).

If $a > b + 1$, we choose m such that $a - b > m > 1$. Since

$$\frac{P_{n+1}}{P_n} = a - b\sqrt[n+1]{c} \quad \text{and} \quad \lim_{n \rightarrow \infty} \sqrt[n+1]{c} = 1,$$

we have $\lim_{n \rightarrow \infty} \frac{P_{n+1}}{P_n} = a - b$. We deduce that for some $N \in \mathbb{N}$, we have $\frac{P_{n+1}}{P_n} > m$ for all $n \geq N$. Then, for any $r \in \mathbb{N}$, we obtain

$$\frac{P_{N+r}}{P_N} = \frac{P_{N+1}}{P_N} \cdot \frac{P_{N+2}}{P_{N+1}} \cdots \frac{P_{N+r}}{P_{N+r-1}} > m^r$$

and therefore $P_{N+r} > m^r \cdot P_N$ for all $r \geq 1$. Since $\lim_{r \rightarrow \infty} m^r = \infty$, we must have $\lim_{n \rightarrow \infty} P_n = \infty$.

If $a < b + 1$, similarly, we choose q such that $a - b < q < 1$ and then for some $N \in \mathbb{N}$, we have $\frac{P_{n+1}}{P_n} < q$ for all $n \geq N$. As above we deduce that $P_{N+r} < q^r P_N$ and so $\lim_{n \rightarrow \infty} P_n = 0$ (since $\lim_{r \rightarrow \infty} q^r = 0$).

Lastly, suppose that $a = b + 1$. Then $a - b\sqrt[n]{c} = 1 - b(\sqrt[n]{c} - 1)$ and when $n \rightarrow \infty$, we have

$$\ln(a - b\sqrt[n]{c}) \sim -b(\sqrt[n]{c} - 1) \sim -\frac{b \ln(c)}{n}$$

(since $\ln(1 - x) \sim -x$ as $x \rightarrow 0$ and $\sqrt[n]{c} - 1 = \exp((\ln(c))/n) - 1 \sim \frac{\ln(c)}{n}$ as $n \rightarrow \infty$).

The series $\sum_{n \geq 2} \ln(a - b\sqrt[n]{c})$ (whose terms are all negative for n large enough) is divergent (since $\sum_{n \geq 2} \frac{1}{n}$ is divergent). This means that $\lim_{n \rightarrow \infty} \ln(P_n) = -\infty$ and so $\lim_{n \rightarrow \infty} P_n = 0$.

4487. Proposed by Martin Lukarevski.

Let a, b, c be the sides of a triangle ABC , m_a, m_b, m_c the corresponding medians and R, r its circumradius and inradius respectively. Prove that

$$\frac{a^2}{m_b^2 + m_c^2} + \frac{b^2}{m_c^2 + m_a^2} + \frac{c^2}{m_a^2 + m_b^2} \geq \frac{4r}{R}.$$

We received 20 correct solutions. We present the solution by Ioannis Sfikas.

By inequality on page 52 of *Geometric Inequalities* by O. Bottema and R. Z. Djordjevic, we have

$$9R^2 \geq a^2 + b^2 + c^2 \quad \text{or} \quad \frac{1}{3}(a^2 + b^2 + c^2) \geq \frac{1}{27R^2}$$

and

$$2s^2 \geq 27Rr \quad \text{or} \quad 2(a + b + c)^2 \geq 108Rr,$$

where s is the semiperimeter of the triangle ABC . Also, we will use the well-known relation

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2).$$

By Titu's lemma (a special case of the Cauchy-Schwarz inequality),

$$\begin{aligned} \frac{a^2}{m_b^2 + m_c^2} + \frac{b^2}{m_c^2 + m_a^2} + \frac{c^2}{m_a^2 + m_b^2} &\geq \frac{(a + b + c)^2}{2(m_a^2 + m_b^2 + m_c^2)} \\ &= \frac{2(a + b + c)^2}{3(a^2 + b^2 + c^2)} \\ &\geq \frac{108Rr}{27R^2} = \frac{4r}{R}. \end{aligned}$$

4488. Proposed by George Apostolopoulos.

Let ABC be an acute-angled triangle. Prove that

$$\sqrt{\cot A} + \sqrt{\cot B} + \sqrt{\cot C} \leq \sqrt{\cot \frac{A}{2} \cdot \cot \frac{B}{2} \cdot \cot \frac{C}{2}}.$$

We received 14 submissions, all correct, and we present the solution by Leonard Giugiuc.

We set $x = \tan \frac{A}{2}$, $y = \tan \frac{B}{2}$, and $z = \tan \frac{C}{2}$. Then $0 < x, y, z < 1$ and it is well known that $xy + yz + zx = 1$.

Since

$$\frac{1 - x^2}{2x} = \frac{1 - \tan^2 A/2}{2 \tan A/2} = \frac{1}{\tan A} = \cot A, \quad \frac{1 - y^2}{2y} = \cot B, \quad \text{and} \quad \frac{1 - z^2}{2z} = \cot C,$$

the given inequality is equivalent to

$$\sum_{cyc} \sqrt{\frac{1-x^2}{2x}} \leq \frac{1}{\sqrt{xyz}} \quad \text{or} \quad \sum_{cyc} \sqrt{\frac{yz(1-x^2)}{2}} \leq 1. \quad (1)$$

By the AM-GM inequality, we have

$$\sqrt{\frac{yz(1-x^2)}{2}} \leq \frac{1}{2} \left(yz + \frac{1-x^2}{2} \right) = \frac{1}{4} (2yz + 1 - x^2).$$

Similarly,

$$\sqrt{\frac{zx(1-y^2)}{2}} \leq \frac{1}{4} (2zx + 1 - y^2),$$

and

$$\sqrt{\frac{xy(1-z^2)}{2}} \leq \frac{1}{4} (2xy + 1 - z^2).$$

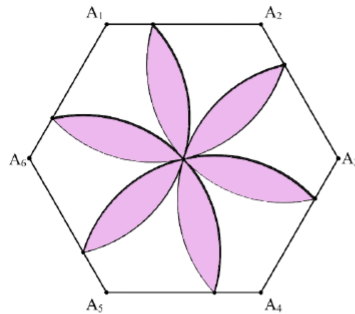
Adding up the three inequalities above then yields

$$\begin{aligned} \sum_{cyc} \sqrt{\frac{yz(1-x^2)}{2}} &\leq \frac{1}{4} (2(xy + yz + zx) + 3 - (x^2 + y^2 + z^2)) \\ &= \frac{1}{4} (5 - (x^2 + y^2 + z^2)) \\ &\leq \frac{1}{4} (5 - (xy + yz + zx)) = 1, \end{aligned}$$

so (1) holds and the proof is complete.

4489. *Proposed by Arsalan Wares.*

Regular hexagon A has its vertices at points A_1, A_2, A_3, A_4, A_5 and A_6 . Six circular congruent arcs are drawn inside hexagon A and all six pass through the center of A . The terminal points of each of the six arcs divide the sides of A in the ratio 3 : 7. The six regions within A that are bounded only by circular arcs have been shaded. Find the ratio of the area of A to the area of the shaded region.



We received 17 correct solutions. We present the solution by Jason Smith.

Let the side length of the hexagon be 10. The area of the entire hexagon is

$$A_{\text{hex}} = 6 \cdot \frac{1}{2} \cdot 10^2 \sin \frac{\pi}{3} = 150\sqrt{3}.$$

Let C denote the center of the hexagon and B_1, B_2, \dots, B_6 the tips of the flower petals. For the squared distance r^2 between consecutive tips, the law of cosines applied to triangle $B_1A_2B_2$ gives

$$r^2 = 3^2 + 7^2 - 2 \cdot 3 \cdot 7 \cdot \cos \frac{2\pi}{3} = 79.$$

We can use this value of r^2 to find the area of the of the two circular segments with vertices B_1 and C . Since the area of one of these circular segments is the area of a sector of the circle minus the area of a triangle, we have

$$A_{\text{seg}} = \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta = \frac{1}{2} \cdot 79 \left(\frac{\pi}{3} - \sin \frac{\pi}{3} \right) = \frac{79}{2} \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right).$$

There are twelve such identical circular segments comprising the shaded region, so

$$A_{\text{sh}} = 12 \cdot \frac{79}{2} \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) = 79(2\pi - 3\sqrt{3}).$$

Therefore, the ratio we seek is equal to

$$\frac{A_{\text{hex}}}{A_{\text{sh}}} = \frac{150\sqrt{3}}{79(2\pi - 3\sqrt{3})} \approx 3.025.$$

4490. Proposed by Borislav Mirchev and Leonard Giugiuc.

A line ℓ through the orthocenter H of the acute triangle ABC meets the circumcircle at points K on the smaller arc AC and L on the smaller arc BC . If M, N , and P are the feet of the perpendiculars to ℓ from the vertices A, B , and C , respectively, prove that $PH = |KM - LN|$.

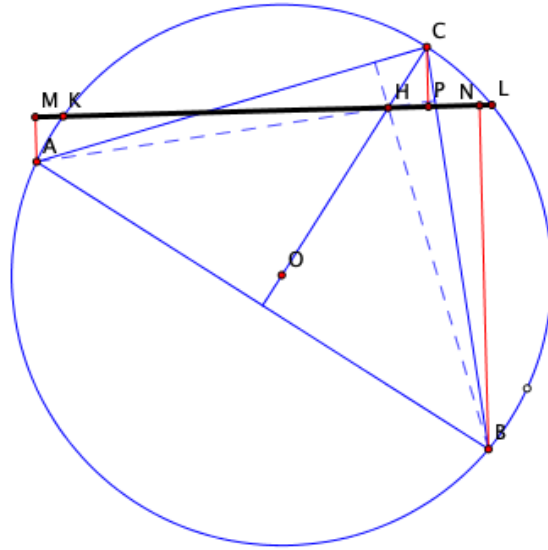
All 7 of the solutions we received were complete and correct, although four of them failed to express clearly the result that they were proving. We feature a composite of the independent solutions by Michel Bataille and by J. Chris Fisher.

As stated, the problem is not correct: see the accompanying figure where $PH = KM + LN$. Moreover, the statement is unnecessarily restrictive. We shall show the following result, which remains true independent of the figure:

A line ℓ through the orthocenter H of an arbitrary triangle ABC meets the circumcircle at points K and L . If M, N , and P are the feet of the perpendiculars to ℓ from the vertices A, B , and C , respectively, then

$$PH = \|\overrightarrow{KM} + \overrightarrow{LN}\|.$$

Therefore, we have that $PH = KM + LN$ if \overrightarrow{KM} and \overrightarrow{LN} have the same sense, and $PH = |KM - LN|$ otherwise.



For the proof, we can assume a unit circumradius, place the circumcenter at the origin, and rotate the figure about O so that ℓ is parallel to the x -axis. If the coordinates of vertices are

$$A(a, a'), B(b, b'), \quad \text{and} \quad C(c, c'),$$

where $a^2 + a'^2 = b^2 + b'^2 = c^2 + c'^2 = 1$, then the orthocenter has coordinates

$$H(a + b + c, a' + b' + c').$$

By assumption, ℓ is the line $y = a' + b' + c'$. The x -coordinates of M, N , and P will therefore be a, b , and c , respectively, while the x -coordinates of K and L have values k and $-k$, which we have no need to calculate. Thus,

$$PH = |(a + b + c) - c| = |a + b|.$$

When M and L are both inside or both outside the circumcircle, the vectors \overrightarrow{KM} and \overrightarrow{LN} point in opposite directions, whence the quantities $k - a$ and $-k - b$ have opposite signs. It follows that

$$|KM - LN| = |k - a + (-k - b)| = |a + b| = PH.$$

When one of M or N is inside the circumcircle and the other outside, then $k - a$ and $-k - b$ have the same sign, and

$$KM + LN = |k - a + (-k - b)| = |a + b| = PH,$$

as claimed.

Editor's comments. When the editors revised the statement of the proposers' problem, their assumption that both M and N lie inside the circumcircle was unfortunately omitted. There seems to be no obvious criteria for predicting the relative behaviour of M and N . It is easy to check that the assumption of acute angles is not sufficient. Consider, for example, an isosceles triangle ABC with $CA = CB$. When $\angle C = 60^\circ$ (and $H = O$) then for all choices of the point K , ℓ is a diameter, and both M and N are on or inside the circumcircle. But should $\angle C > 60^\circ$ (and H be between C and O as in the above figure) then K can be chosen sufficiently close to A on the arc AC that omits B so that M is outside the circumcircle while N is inside. (This serves as a counterexample to the problem as revised by the editors.) Note that the value of the y -coordinate plays no role in the argument so that an analogous result holds for lines parallel to ℓ . Note, finally, that when $\angle C = 90^\circ$, we have a rather nice, easily proved result, namely $KM = CN$ (which is a consequence of our result for right triangles since $PH = 0$ and $C = L$).

