OLYMPIAD CORNER

No. 382

The problems in this section appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **June 15, 2020**.

OC476. Let x be a real number such that both sums $S = \sin 64x + \sin 65x$ and $C = \cos 64x + \cos 65x$ are rational numbers. Prove that in one of these sums, both terms are rational.

OC477. Let $A = \{z \in \mathbb{C} \mid |z| = 1\}$.

(a) Prove that $(|z+1| - \sqrt{2})(|z-1| - \sqrt{2}) \leq 0 \ \forall z \in A$.

(b) Prove that for any $z_1, z_2, \ldots, z_{12} \in A$, there is a choice of signs "$\pm$" so that

$$\sum_{k=1}^{12} |z_k \pm 1| < 17.$$

OC478. Consider two noncommuting matrices $A, B \in \mathcal{M}_2(\mathbb{R})$.

(a) Knowing that $A^3 = B^3$, prove that A^n and B^n have the same trace for any nonzero natural number n.

(b) Give an example of two noncommuting matrices $A, B \in \mathcal{M}_2(\mathbb{R})$ such that for any nonzero $n \in \mathbb{N}$, $A^n \neq B^n$, and A^n and B^n have different trace.

OC479. We say that the function $f : \mathbb{Q}_+^* \to \mathbb{Q}$ has the property $\mathcal{P}$ if

$$f(xy) = f(x) + f(y) \qquad \forall x, y \in \mathbb{Q}_+^*.$$

(a) Prove that there do not exist injective functions with property $\mathcal{P}$.

(b) Do there exist surjective functions with property $\mathcal{P}$?

OC480. In the plane, there are points C and D on the same region with respect to the line defined by the segment AB so that the circumcircles of triangles ABC and ABD are the same. Let E be the incenter of triangle ABC, let F be the incenter of triangle ABD and let G be the midpoint of the arc AB not containing the points C and D. Prove that points A, B, E, F are on a circle with center G.

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC476. Soit x un nombre réel tel que les deux sommes $S = \sin 64x + \sin 65x$ et $C = \cos 64x + \cos 65x$ sont rationnelles. Démontrer que dans une des sommes, les deux termes sont rationnels.

OC477. Soit $A = \{z \in \mathbb{C} \mid |z| = 1\}$.

(a) Démontrer que $(|z+1| - \sqrt{2})(|z-1| - \sqrt{2}) \leq 0 \; \forall z \in A$.

(b) Démontrer que pour tout $z_1, z_2, \ldots, z_{12} \in A$, il existe un choix de signes "$\pm$" tels que

$$\sum_{k=1}^{12} |z_k \pm 1| < 17.$$

OC478. Soient deux matrices qui ne commutent pas, $A, B \in \mathcal{M}_2(\mathbb{R})$.

(a) Si $A^3 = B^3$, démontrer que A^n et B^n ont la même trace $\forall n \in \mathbb{N}$, $n \neq 0$.

(b) Donner une exemple de deux matrices qui ne commutent pas, $A, B \in \mathcal{M}_2(\mathbb{R})$, telles que, pour tout non nul $n \in \mathbb{N}$, $A^n \neq B^n$, puis A^n et B^n sont de différentes traces.

OC479. La fonction $f : \mathbb{Q}_+^* \to \mathbb{Q}$ possède la propriété $\mathcal{P}$ si

$$f(xy) = f(x) + f(y) \qquad \forall x, y \in \mathbb{Q}_+^*.$$

(a) Démontrer qu'il n'existe aucune fonction injective possédant la propriété $\mathcal{P}$.

(b) Des fonctions surjectives avec la propriété $\mathcal{P}$ existent-elles?

OC480. Soient C et D deux points dans le même demi plan par rapport au segment AB, de façon à ce que les cercles circonscrits des triangles ABC et ABD soient les mêmes. Soit E le centre du cercle inscrit du triangle ABC et soit F le centre du cercle inscrit du triangle ABD ; soit aussi G le mi point de l'arc AB contenant ni C ni D. Démontrer que A, B, E, F se trouvent sur un cercle de centre G.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(10), p. 504–505.

OC451. Determine the least natural number a such that

$$a \geq \sum_{k=1}^{n} a_k \cos(a_1 + \cdots + a_k)$$

for any nonzero natural number n and for any positive real numbers $a_1, a_2, \ldots, a_n$ whose sum is at most π.

We received 1 submission. We present the solution by Oliver Geupel.

We show that

$$\sup \left\{ \sum_{k=1}^{n} a_k \cos(a_1 + \cdots + a_k) \ : \ n \geq 1;\ a_1, \ldots, a_k > 0;\ \sum_{k=1}^{n} a_k \leq \pi \right\} = 1,$$

which implies that the least value of a is 1. For $1 \leq k \leq n$, let

$$x_k = \sum_{j=1}^{k} a_j.$$

Then, $0 = x_0 < x_1 < x_2 < \cdots < x_n \leq \pi$, and the sum

$$\sum_{k=1}^{n} a_k \cos(a_1 + \cdots + a_k) = \sum_{k=1}^{n} (x_k - x_{k-1}) \cos x_k \qquad (1)$$

is a right Riemann sum which underestimates the integral

$$I(x_n) = \int_0^{x_n} \cos x \, \mathrm{d}x$$

of the decreasing function $\cos x$ on the interval $[0, x_n]$. Since $I(x_n) = \sin x_n \leq 1$ for every $x_n \leq \pi$, we obtain that $a \leq 1$. However, if $a_k = \pi/(2n)$ and $x_k = k\pi/(2n)$ for any $1 \leq k \leq n$ then the sequence defined by the sums (1) converges towards $I(\pi/2) = 1$ as $n \to \infty$. Hence $a = 1$.

Editor's Comment. The restrictions on a_n's can be changed. For example, if

$$a_1 + \cdots + a_n \leq \pi/2,$$

then the value of the upper bound a is 1, as before. However, if

$$a_1 + \cdots + a_n = \pi,$$

then $a = 0$.

OC452. Let $ABCD$ be a square. Consider the points $E \in AB$, $N \in CD$ and $F, M \in BC$ such that triangles AMN and DEF are equilateral. Prove that $PQ = FM$, where $\{P\} = AN \cap DE$ and $\{Q\} = AM \cap EF$.

We received 11 *correct submissions. We present two solutions.*

Solution 1, by Miguel Amengual Covas.

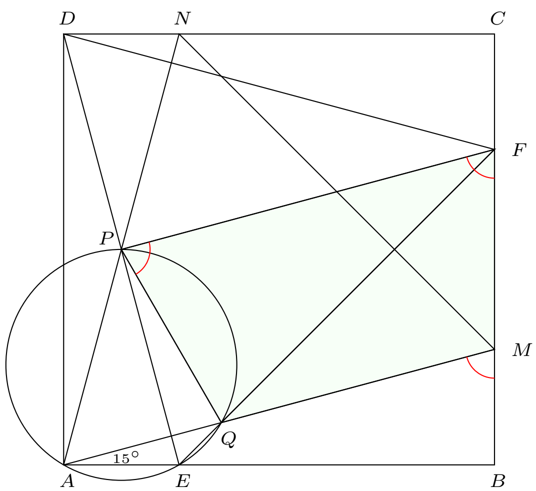
The right-angled triangles ABM and ADN have equal hypotenuses AM and AN, and the legs AB and AD are respectively equal. Thus $\triangle ABM$ and $\triangle ADN$ are congruent with $\angle MAB = \angle NAD$.

Now, $\angle DAB = \angle MAB + \angle NAM + \angle NAD$. Next, since $\angle DAB = 90°$ and $\angle NAM = 60°$, it follows that $\angle MAB$ and $\angle NAD$ are each $15°$. Analogously, $\angle CDF = \angle ADE = 15°$.

Clearly, then, $\triangle ABM$, $\triangle CDF$, $\triangle DAE$, $\triangle ADN$ are congruent (these are right-angled triangles which have equal legs AB, CD and DA and contain another pair of equal angles) with

$$AE = BM = FC = ND.$$

Consequently, $AEND$ is a rectangle, so that the segments AN and DE bisect each other. Thus P is the midpoint of segment DE. Then, we have in equilateral triangle DEF that $\angle FPE$ is a right angle and $\angle PFE = 30°$.



Subtracting $AE = FC$ from both sides of $AB = BC$ gives $AB - AE = BC - FC$. This makes $EB = BF$ and consequently $\triangle EBF$ is an isosceles right-triangle with $\angle EFB = 45°$. Therefore,

$$\begin{aligned} \angle PFM = \angle PFB = \angle PFE + \angle EFB &= 30° + 45° \\ &= 75° \\ &= 90° - 15° \\ &= 90° - \angle MAB \\ &= \angle AMB, \end{aligned}$$

implying that PF is parallel to AM, that is, $PF \parallel QM$.

Next, PQ subtends 60° angles at A and E, making $PAEQ$ cyclic and on chord EQ we have

$$\angle QPE = \angle QAE = \angle MAB = 15^\circ$$

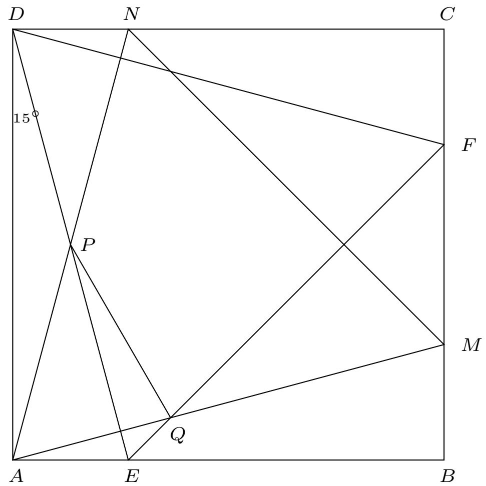
and

$$\angle FPQ = \angle FPE - \angle QPE = 90^\circ - 15^\circ = 75^\circ.$$

That is to say, the exterior angle QMB in quadrilateral $PQMF$ is equal to the interior and opposite angle P. Thus $PQMF$ is cyclic. Since $PF \parallel QM$, $PQMF$ is an isosceles trapezium. The conclusion follows.

Solution 2, by Miguel Amengual Covas.

As in Solution 1, we conclude that $\angle MAB = \angle NAD = \angle CDF = \angle ADE = 15^\circ$.



Suppose (wlog) the unity of measurement equal to the length of the side of the given square. Then

$$DN = AE = BM = CF = \tan 15^\circ. \tag{1}$$

Hence,

$$FM = BC - CF - MB = 1 - 2\tan 15^\circ. \tag{2}$$

Moreover, since $AE \parallel DN$, $AEND$ is a rectangle, so that segments AN and DE bisect each other. Therefore, P is the midpoint of DE and we have

$$PE = \frac{1}{2}DE = \frac{1}{2\cos 15^\circ}. \tag{3}$$

Observing that the equal segments FB and BE make EBF an isosceles right-angled triangle, the exterior angle theorem, applied to $\triangle AEQ$ at E, yields

$$\angle EQA = \angle QEB - \angle QAE = \angle FEB - \angle MAB = 45^\circ - 15^\circ = 30^\circ.$$

Now, the law of sines asserts that

$$\frac{QE}{\sin 15^\circ} = \frac{AE}{\sin 30^\circ},$$

and therefore

$$\frac{QE}{\sin 15^\circ} = 2 \cdot AE,$$

yielding (by (2))

$$QE = \frac{2\sin^2 15^\circ}{\cos 15^\circ}. \tag{4}$$

Applying the law of cosines to $\triangle PEQ$ we get

$$PQ^2 = PE^2 + QE^2 - 2 \cdot PE \cdot QE \cdot \cos 60^\circ.$$

Substituting for PE and QE from (4) and (5),

$$PQ^2 = \frac{1}{4\cos^2 15^\circ} + \frac{4\sin^4 15^\circ}{\cos^2 15^\circ} - \tan^2 15^\circ. \tag{5}$$

Now, we write the identity $2\sin 30^\circ = 1$ in the equivalent form $4\sin 15^\circ \cos 15^\circ = 1$, multiply it by $\tan 15^\circ$ and square, obtaining $16\sin^4 15^\circ = \tan^2 15^\circ$. Hence we can rewrite (6) as

$$PQ^2 = \frac{1 + \tan^2 15^\circ}{4\cos^2 15^\circ} - \tan^2 15^\circ,$$

or, equivalently,

$$PQ^2 = \frac{1}{4\cos^4 15^\circ} - \tan^2 15^\circ.$$

This, in turn, is equivalent to

$$PQ^2 = \frac{1 - 4\sin^2 15^\circ \cos^2 15^\circ}{4\cos^4 15^\circ} = \frac{1 - \sin^2 30^\circ}{4\cos^4 15^\circ} = \frac{3}{16\cos^4 15^\circ}.$$

Thus

$$PQ = \frac{\sqrt{3}}{4\cos^2 15^\circ} = \sqrt{3}\tan 15^\circ. \tag{6}$$

Taking into account that $\tan 15^\circ = 2 - \sqrt{3}$, from (3) and (7) we conclude that

$$FM = 2\sqrt{3} - 3 = PQ.$$

OC453. Let $n \geq 2$ be an integer and let $A, B \in \mathcal{M}_n(\mathbb{C})$. If $(AB)^3 = O_n$, is it true that $(BA)^3 = O_n$? Justify your answer.

We received 4 correct submissions. We present two solutions.

Solution 1, by Oliver Geupel.

We prove that the deduction is correct if and only if $n \leq 3$.

First, let $n \leq 3$ and let λ be an eigenvalue of the matrix $C = BA$ with eigenvector v. Then, $\lambda v = Cv$ and

$$\lambda^4 v = C \cdot \lambda^3 v = C^2 \cdot \lambda^2 v = C^3 \cdot \lambda v = C^4 v = B(AB)^3 Av = BO_n Av = O_n.$$

Hence $\lambda = 0$. So 0 is the only eigenvalue of C. The characteristic polynomial of C is then λ^n. By the Cayley-Hamilton theorem, the matrix C satisfies its own characteristic equation, so that $C^n = O_n$ and therefore $(BA)^3 = C^3 = O_n$.

We now turn to the case where $n \geq 4$. Let

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consider the n-by-n block matrices

$$A = \begin{bmatrix} A_4 & O_{4\times(n-4)} \\ O_{(n-4)\times 4} & O_{(n-4)\times(n-4)} \end{bmatrix}, \qquad B = \begin{bmatrix} B_4 & O_{4\times(n-4)} \\ O_{(n-4)\times 4} & O_{(n-4)\times(n-4)} \end{bmatrix}.$$

Straightforward computations yield $(A_4B_4)^3 = O_4$ and

$$(B_4A_4)^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus,

$$(AB)^3 = \begin{bmatrix} (A_4B_4)^3 & O_{4\times(n-4)} \\ O_{(n-4)\times 4} & O_{(n-4)\times(n-4)} \end{bmatrix} = O_n$$

and

$$(BA)^3 = \begin{bmatrix} (B_4A_4)^3 & O_{4\times(n-4)} \\ O_{(n-4)\times 4} & O_{(n-4)\times(n-4)} \end{bmatrix} \neq O_n.$$

This completes the proof.

Solution 2, by Missouri State University Problem Solving Group.

We prove a more general result. Fix $k \geq 1$. Then

(1) If $n \leq k$ then $(AB)^k = 0$ implies $(BA)^k = 0$.

(2) If $n \geq k+1$ then there exist $n \times n$ matrices A and B such that $(AB)^k = 0$ but $(BA)^k \neq 0$.

Suppose $n \leq k$ and $(AB)^k = 0$. Every eigenvalue of AB is 0 and the characteristic polynomial of AB is x^n. But AB and BA have the same characteristic polynomial, hence $(BA)^k = 0$. This proves (1).

(2) Let $n = k+1$, let A be the $n \times n$ matrix whose (i, j) entry is 1 if and only if $i = j > 1$ and 0 otherwise, and let B be the $n \times n$ matrix whose (i, j) entry is 1 if and only if $j = i+1$ and 0 otherwise. Then $BA = B$ and $AB = C$ where the (i, j) entry of C is 1 if and only if $j = i + 1 > 2$ and 0 otherwise.

A direct calculation yields $(AB)^{n-1} = 0$, but $(BA)^{n-1}$ is non-zero. By taking the direct sum of A and B with the zero matrix, we can get counterexamples for all larger n.

For the original problem with $k = 3$, we take $n = 4$ and get

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (AB)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

but

$$BA = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (BA)^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

OC454. Find all the functions $f : \mathbb{N} \to \mathbb{N}$ having the following property for each natural number m: if $d_1, d_2, \ldots, d_n$ are all the divisors of the number m, then

$$f(d_1)f(d_2) \cdot \ldots \cdot f(d_n) = m.$$

We received 8 submissions. We present the solution by Oliver Geupel.

It is readily checked that the following function is a solution to the problem:

$$f(m) = \begin{cases} p & \text{if } m = p^k \text{ where } p \text{ is a prime number and } k \geq 1 \\ 1 & \text{otherwise.} \end{cases}$$

We show that there are no other solutions. Suppose f is any solution.

Putting $m = 1$ in the given condition, we obtain $f(1) = 1$. Setting $m = p$ where p is a prime number in the given condition, we get $f(p) = f(1)f(p) = p$. A straightforward induction shows that $f(p^k) = p$ for $k \geq 1$.

Finally, let m have at least two distinct prime divisors, say $m = p_1^{k_1} p_2^{k_2} \cdots p_\ell^{k_\ell}$, where $p_1, \ldots, p_\ell$ are distinct prime divisors ($\ell \geq 2$) and $k_j \geq 1$ for $1 \leq j \leq \ell$. Let D denote the set of those divisors of m that have at least two distinct prime divisors. Then,

$$m = f(1) \left(\prod_{\substack{1 \leq j \leq \ell \\ 1 \leq i \leq k_j}} f(p_j^i) \right) \left(\prod_{d \in D} f(d) \right) = \left(\prod_{1 \leq j \leq \ell} p_j^{k_j} \right) \left(\prod_{d \in D} f(d) \right) = m \prod_{d \in D} f(d).$$

Hence,

$$\prod_{d \in D} f(d) = 1.$$

It follows $f(d) = 1$ for all $d \in D$. Since $m \in D$, we conclude that $f(m) = 1$.

OC455. Let D be a point on the base AB of an isosceles triangle ABC. Select a point E so that $ADEC$ is a parallelogram. On the line ED, take a point F such that $E \in DF$ and $EB = EF$. Prove that the length of the chord that the line BE cuts on the circumcircle of triangle ABF is twice the length of the segment AC.

We received 4 *submissions. We present the solution by Ivko Dimitrić.*

Without loss of generality, we may assume that the vertices are labeled counterclockwise and that $|DB| \leq |AD|$. Let ω be the circumcircle of $\triangle ABF$, O its center and K the point where the line BE meets ω again. Let M be the midpoint of BK, S the midpoint of BF and $N = \overleftrightarrow{CE} \cap \overleftrightarrow{BF}$. Further, set

$$\alpha = \angle CAB = \angle CBA = \angle ECB, \quad \theta = \angle FBE = \angle EFB \quad \text{and} \quad \varphi = \angle ECM.$$

To prove the claim, it suffices to show that $\triangle CBM$ is isosceles.

From parallelogram $ADEC$ we have $\angle CED = \alpha$ and from isosceles $\triangle BFE$ we get $\angle DEB = 2\theta$, so that $\angle CEB = \alpha + 2\theta$. The points O, E and S are collinear, because $BE = EF$ and E, O belong to the perpendicular bisector of BF at S. Since $|DB| \leq |AD|$ the foot of the perpendicular from B to CN belongs to the segment $\overline{CE}$ just as the foot of the perpendicular from C to AB belongs to $\overline{AD}$.

That implies that $\angle BEN \geq 90^\circ$ and $\angle MEO = \angle BES < 90^\circ$. Hence, S is between B and N and M is between E and K.

Since $OM \perp BK$ and O and C belong to the perpendicular bisector of AB we have $\angle ECO = \angle OME = 90^\circ$ and the quadrilateral $CEMO$ is cyclic so that

$$\angle ECM = \angle EOM = \varphi \quad \text{and} \quad \angle OEC = \angle OMC.$$

Since the triangles ESN and OEM are right-angled we have

$$\begin{aligned} \angle CNB = \angle ENS &= 90^\circ - \angle SEN \\ &= 90^\circ - \angle OEC \\ &= 90^\circ - \angle OMC \\ &= \angle CME = \angle CMB. \end{aligned}$$

Therefore, quadrilateral $BCMN$ is cyclic so that $\varphi = \angle NCM = \angle NBM = \theta$. Now, we have

$$\angle BCM = \angle BCE + \angle ECM = \alpha + \varphi$$

and from $\triangle CME$ we get

$$\angle CMB = \angle CEB - \angle ECM = (\alpha + 2\theta) - \varphi = \alpha + 2\varphi - \varphi = \alpha + \varphi.$$

Therefore, we conclude that $\angle BCM = \angle CMB$, so that

$$AC = CB = MB = \frac{1}{2}BK,$$

proving the claim.