OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(10), p. 504–505.

OC451. Determine the least natural number a such that

$$a \geq \sum_{k=1}^{n} a_k \cos(a_1 + \cdots + a_k)$$

for any nonzero natural number n and for any positive real numbers $a_1, a_2, \ldots, a_n$ whose sum is at most π.

We received 1 submission. We present the solution by Oliver Geupel.

We show that

$$\sup\left\{\sum_{k=1}^{n} a_k \cos(a_1 + \cdots + a_k) \;:\; n \geq 1;\; a_1, \ldots, a_k > 0;\; \sum_{k=1}^{n} a_k \leq \pi\right\} = 1,$$

which implies that the least value of a is 1. For $1 \leq k \leq n$, let

$$x_k = \sum_{j=1}^{k} a_j.$$

Then, $0 = x_0 < x_1 < x_2 < \cdots < x_n \leq \pi$, and the sum

$$\sum_{k=1}^{n} a_k \cos(a_1 + \cdots + a_k) = \sum_{k=1}^{n} (x_k - x_{k-1}) \cos x_k \tag{1}$$

is a right Riemann sum which underestimates the integral

$$I(x_n) = \int_0^{x_n} \cos x \,\mathrm{d}x$$

of the decreasing function $\cos x$ on the interval $[0, x_n]$. Since $I(x_n) = \sin x_n \leq 1$ for every $x_n \leq \pi$, we obtain that $a \leq 1$. However, if $a_k = \pi/(2n)$ and $x_k = k\pi/(2n)$ for any $1 \leq k \leq n$ then the sequence defined by the sums (1) converges towards $I(\pi/2) = 1$ as $n \to \infty$. Hence $a = 1$.

Editor's Comment. The restrictions on a_n's can be changed. For example, if

$$a_1 + \cdots + a_n \leq \pi/2,$$

then the value of the upper bound a is 1, as before. However, if

$$a_1 + \cdots + a_n = \pi,$$

then $a = 0$.

OC452. Let $ABCD$ be a square. Consider the points $E \in AB$, $N \in CD$ and $F, M \in BC$ such that triangles AMN and DEF are equilateral. Prove that $PQ = FM$, where $\{P\} = AN \cap DE$ and $\{Q\} = AM \cap EF$.

We received 11 *correct submissions. We present two solutions.*

Solution 1, by Miguel Amengual Covas.

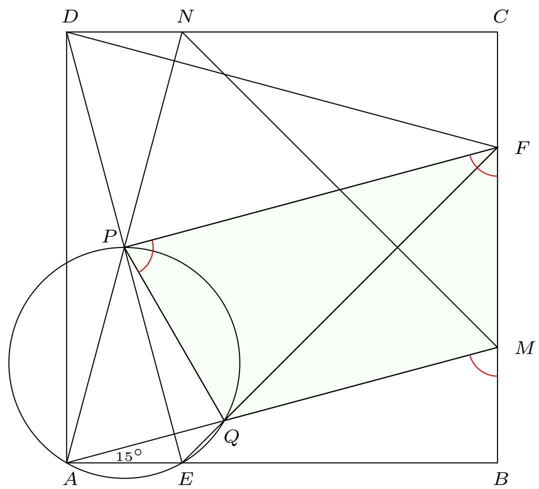
The right-angled triangles ABM and ADN have equal hypotenuses AM and AN, and the legs AB and AD are respectively equal. Thus $\triangle ABM$ and $\triangle ADN$ are congruent with $\angle MAB = \angle NAD$.

Now, $\angle DAB = \angle MAB + \angle NAM + \angle NAD$. Next, since $\angle DAB = 90°$ and $\angle NAM = 60°$, it follows that $\angle MAB$ and $\angle NAD$ are each $15°$. Analogously, $\angle CDF = \angle ADE = 15°$.

Clearly, then, $\triangle ABM$, $\triangle CDF$, $\triangle DAE$, $\triangle ADN$ are congruent (these are right-angled triangles which have equal legs AB, CD and DA and contain another pair of equal angles) with

$$AE = BM = FC = ND.$$

Consequently, $AEND$ is a rectangle, so that the segments AN and DE bisect each other. Thus P is the midpoint of segment DE. Then, we have in equilateral triangle DEF that $\angle FPE$ is a right angle and $\angle PFE = 30°$.



Subtracting $AE = FC$ from both sides of $AB = BC$ gives $AB - AE = BC - FC$. This makes $EB = BF$ and consequently $\triangle EBF$ is an isosceles right-triangle with $\angle EFB = 45°$. Therefore,

$$\begin{aligned} \angle PFM = \angle PFB = \angle PFE + \angle EFB &= 30° + 45° \\ &= 75° \\ &= 90° - 15° \\ &= 90° - \angle MAB \\ &= \angle AMB, \end{aligned}$$

implying that PF is parallel to AM, that is, $PF \parallel QM$.

Next, PQ subtends 60° angles at A and E, making $PAEQ$ cyclic and on chord EQ we have

$$\angle QPE = \angle QAE = \angle MAB = 15^\circ$$

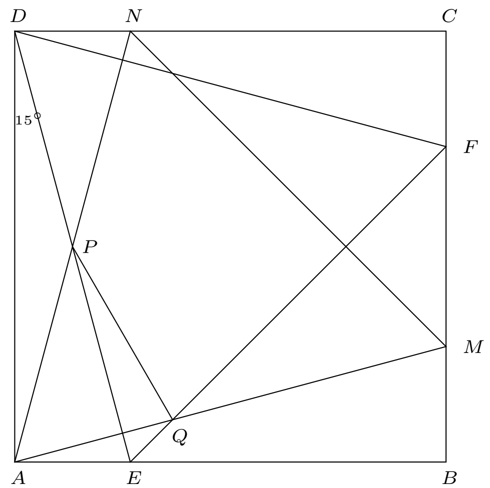
and

$$\angle FPQ = \angle FPE - \angle QPE = 90^\circ - 15^\circ = 75^\circ.$$

That is to say, the exterior angle QMB in quadrilateral $PQMF$ is equal to the interior and opposite angle P. Thus $PQMF$ is cyclic. Since $PF \parallel QM$, $PQMF$ is an isosceles trapezium. The conclusion follows.

Solution 2, by Miguel Amengual Covas.

As in Solution 1, we conclude that $\angle MAB = \angle NAD = \angle CDF = \angle ADE = 15^\circ$.



Suppose (wlog) the unity of measurement equal to the length of the side of the given square. Then

$$DN = AE = BM = CF = \tan 15^\circ. \tag{1}$$

Hence,

$$FM = BC - CF - MB = 1 - 2\tan 15^\circ. \tag{2}$$

Moreover, since $AE \parallel DN$, $AEND$ is a rectangle, so that segments AN and DE bisect each other. Therefore, P is the midpoint of DE and we have

$$PE = \frac{1}{2}DE = \frac{1}{2\cos 15^\circ}. \tag{3}$$

Observing that the equal segments FB and BE make EBF an isosceles right-angled triangle, the exterior angle theorem, applied to $\triangle AEQ$ at E, yields

$$\angle EQA = \angle QEB - \angle QAE = \angle FEB - \angle MAB = 45^\circ - 15^\circ = 30^\circ.$$

Now, the law of sines asserts that

$$\frac{QE}{\sin 15^\circ} = \frac{AE}{\sin 30^\circ},$$

and therefore

$$\frac{QE}{\sin 15^\circ} = 2 \cdot AE,$$

yielding (by (2))

$$QE = \frac{2\sin^2 15^\circ}{\cos 15^\circ}. \tag{4}$$

Applying the law of cosines to $\triangle PEQ$ we get

$$PQ^2 = PE^2 + QE^2 - 2 \cdot PE \cdot QE \cdot \cos 60^\circ.$$

Substituting for PE and QE from (4) and (5),

$$PQ^2 = \frac{1}{4\cos^2 15^\circ} + \frac{4\sin^4 15^\circ}{\cos^2 15^\circ} - \tan^2 15^\circ. \tag{5}$$

Now, we write the identity $2\sin 30^\circ = 1$ in the equivalent form $4\sin 15^\circ \cos 15^\circ = 1$, multiply it by $\tan 15^\circ$ and square, obtaining $16\sin^4 15^\circ = \tan^2 15^\circ$. Hence we can rewrite (6) as

$$PQ^2 = \frac{1 + \tan^2 15^\circ}{4\cos^2 15^\circ} - \tan^2 15^\circ,$$

or, equivalently,

$$PQ^2 = \frac{1}{4\cos^4 15^\circ} - \tan^2 15^\circ.$$

This, in turn, is equivalent to

$$PQ^2 = \frac{1 - 4\sin^2 15^\circ \cos^2 15^\circ}{4\cos^4 15^\circ} = \frac{1 - \sin^2 30^\circ}{4\cos^4 15^\circ} = \frac{3}{16\cos^4 15^\circ}.$$

Thus

$$PQ = \frac{\sqrt{3}}{4\cos^2 15^\circ} = \sqrt{3}\tan 15^\circ. \tag{6}$$

Taking into account that $\tan 15^\circ = 2 - \sqrt{3}$, from (3) and (7) we conclude that

$$FM = 2\sqrt{3} - 3 = PQ.$$

OC453. Let $n \geq 2$ be an integer and let $A, B \in \mathcal{M}_n(\mathbb{C})$. If $(AB)^3 = O_n$, is it true that $(BA)^3 = O_n$? Justify your answer.

We received 4 correct submissions. We present two solutions.

Solution 1, by Oliver Geupel.

We prove that the deduction is correct if and only if $n \leq 3$.

First, let $n \leq 3$ and let λ be an eigenvalue of the matrix $C = BA$ with eigenvector v. Then, $\lambda v = Cv$ and

$$\lambda^4 v = C \cdot \lambda^3 v = C^2 \cdot \lambda^2 v = C^3 \cdot \lambda v = C^4 v = B(AB)^3 Av = BO_n Av = O_n.$$

Hence $\lambda = 0$. So 0 is the only eigenvalue of C. The characteristic polynomial of C is then λ^n. By the Cayley-Hamilton theorem, the matrix C satisfies its own characteristic equation, so that $C^n = O_n$ and therefore $(BA)^3 = C^3 = O_n$.

We now turn to the case where $n \geq 4$. Let

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consider the n-by-n block matrices

$$A = \begin{bmatrix} A_4 & O_{4\times(n-4)} \\ O_{(n-4)\times 4} & O_{(n-4)\times(n-4)} \end{bmatrix}, \qquad B = \begin{bmatrix} B_4 & O_{4\times(n-4)} \\ O_{(n-4)\times 4} & O_{(n-4)\times(n-4)} \end{bmatrix}.$$

Straightforward computations yield $(A_4B_4)^3 = O_4$ and

$$(B_4A_4)^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus,

$$(AB)^3 = \begin{bmatrix} (A_4B_4)^3 & O_{4\times(n-4)} \\ O_{(n-4)\times 4} & O_{(n-4)\times(n-4)} \end{bmatrix} = O_n$$

and

$$(BA)^3 = \begin{bmatrix} (B_4A_4)^3 & O_{4\times(n-4)} \\ O_{(n-4)\times 4} & O_{(n-4)\times(n-4)} \end{bmatrix} \neq O_n.$$

This completes the proof.

Solution 2, by Missouri State University Problem Solving Group.

We prove a more general result. Fix $k \geq 1$. Then

(1) If $n \leq k$ then $(AB)^k = 0$ implies $(BA)^k = 0$.

(2) If $n \geq k+1$ then there exist $n \times n$ matrices A and B such that $(AB)^k = 0$ but $(BA)^k \neq 0$.

Suppose $n \leq k$ and $(AB)^k = 0$. Every eigenvalue of AB is 0 and the characteristic polynomial of AB is x^n. But AB and BA have the same characteristic polynomial, hence $(BA)^k = 0$. This proves (1).

(2) Let $n = k+1$, let A be the $n \times n$ matrix whose (i, j) entry is 1 if and only if $i = j > 1$ and 0 otherwise, and let B be the $n \times n$ matrix whose (i, j) entry is 1 if and only if $j = i + 1$ and 0 otherwise. Then $BA = B$ and $AB = C$ where the (i, j) entry of C is 1 if and only if $j = i + 1 > 2$ and 0 otherwise.

A direct calculation yields $(AB)^{n-1} = 0$, but $(BA)^{n-1}$ is non-zero. By taking the direct sum of A and B with the zero matrix, we can get counterexamples for all larger n.

For the original problem with $k = 3$, we take $n = 4$ and get

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (AB)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

but

$$BA = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (BA)^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

OC454. Find all the functions $f : \mathbb{N} \to \mathbb{N}$ having the following property for each natural number m: if $d_1, d_2, \ldots, d_n$ are all the divisors of the number m, then

$$f(d_1)f(d_2) \cdot \ldots \cdot f(d_n) = m.$$

We received 8 submissions. We present the solution by Oliver Geupel.

It is readily checked that the following function is a solution to the problem:

$$f(m) = \begin{cases} p & \text{if } m = p^k \text{ where } p \text{ is a prime number and } k \geq 1 \\ 1 & \text{otherwise.} \end{cases}$$

We show that there are no other solutions. Suppose f is any solution.

Putting $m = 1$ in the given condition, we obtain $f(1) = 1$. Setting $m = p$ where p is a prime number in the given condition, we get $f(p) = f(1)f(p) = p$. A straightforward induction shows that $f(p^k) = p$ for $k \geq 1$.

Finally, let m have at least two distinct prime divisors, say $m = p_1^{k_1} p_2^{k_2} \cdots p_\ell^{k_\ell}$, where $p_1, \ldots, p_\ell$ are distinct prime divisors ($\ell \geq 2$) and $k_j \geq 1$ for $1 \leq j \leq \ell$. Let D denote the set of those divisors of m that have at least two distinct prime divisors. Then,

$$m = f(1) \left(\prod_{\substack{1 \leq j \leq \ell \\ 1 \leq i \leq k_j}} f(p_j^i) \right) \left(\prod_{d \in D} f(d) \right) = \left(\prod_{1 \leq j \leq \ell} p_j^{k_j} \right) \left(\prod_{d \in D} f(d) \right) = m \prod_{d \in D} f(d).$$

Hence,

$$\prod_{d \in D} f(d) = 1.$$

It follows $f(d) = 1$ for all $d \in D$. Since $m \in D$, we conclude that $f(m) = 1$.

OC455. Let D be a point on the base AB of an isosceles triangle ABC. Select a point E so that $ADEC$ is a parallelogram. On the line ED, take a point F such that $E \in DF$ and $EB = EF$. Prove that the length of the chord that the line BE cuts on the circumcircle of triangle ABF is twice the length of the segment AC.

We received 4 submissions. We present the solution by Ivko Dimitrić.

Without loss of generality, we may assume that the vertices are labeled counterclockwise and that $|DB| \leq |AD|$. Let ω be the circumcircle of $\triangle ABF$, O its center and K the point where the line BE meets ω again. Let M be the midpoint of BK, S the midpoint of BF and $N = \overleftrightarrow{CE} \cap \overleftrightarrow{BF}$. Further, set

$$\alpha = \angle CAB = \angle CBA = \angle ECB, \quad \theta = \angle FBE = \angle EFB \quad \text{and} \quad \varphi = \angle ECM.$$

To prove the claim, it suffices to show that $\triangle CBM$ is isosceles.

From parallelogram $ADEC$ we have $\angle CED = \alpha$ and from isosceles $\triangle BFE$ we get $\angle DEB = 2\theta$, so that $\angle CEB = \alpha + 2\theta$. The points O, E and S are collinear, because $BE = EF$ and E, O belong to the perpendicular bisector of BF at S. Since $|DB| \leq |AD|$ the foot of the perpendicular from B to CN belongs to the segment $\overline{CE}$ just as the foot of the perpendicular from C to AB belongs to $\overline{AD}$.

That implies that $\angle BEN \geq 90^\circ$ and $\angle MEO = \angle BES < 90^\circ$. Hence, S is between B and N and M is between E and K.

Since $OM \perp BK$ and O and C belong to the perpendicular bisector of AB we have $\angle ECO = \angle OME = 90^\circ$ and the quadrilateral $CEMO$ is cyclic so that

$$\angle ECM = \angle EOM = \varphi \quad \text{and} \quad \angle OEC = \angle OMC.$$

Since the triangles ESN and OEM are right-angled we have

$$\begin{aligned}\angle CNB = \angle ENS &= 90^\circ - \angle SEN \\ &= 90^\circ - \angle OEC \\ &= 90^\circ - \angle OMC \\ &= \angle CME = \angle CMB.\end{aligned}$$

Therefore, quadrilateral $BCMN$ is cyclic so that $\varphi = \angle NCM = \angle NBM = \theta$. Now, we have

$$\angle BCM = \angle BCE + \angle ECM = \alpha + \varphi$$

and from $\triangle CME$ we get

$$\angle CMB = \angle CEB - \angle ECM = (\alpha + 2\theta) - \varphi = \alpha + 2\varphi - \varphi = \alpha + \varphi.$$

Therefore, we conclude that $\angle BCM = \angle CMB$, so that

$$AC = CB = MB = \frac{1}{2}BK,$$

proving the claim.