MathemAttic

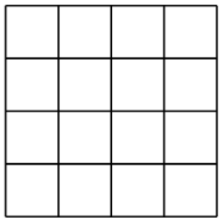
No. 14

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

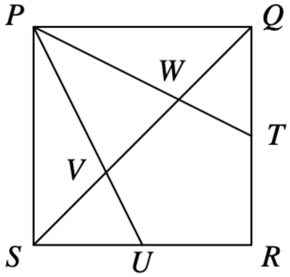
To facilitate their consideration, solutions should be received by **June 15, 2020**.

MA66. The 16 small squares shown in the diagram each have a side length of 1 unit. How many pairs of vertices (intersections of lines) are there in the diagram whose distance apart is an integer number of units?

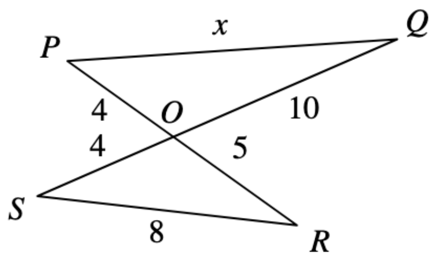


MA67. Consider numbers of the form $10n + 1$, where n is a positive integer. We shall call such a number *grime* if it cannot be expressed as the product of two smaller numbers, possibly equal, both of which are of the form $10k + 1$, where k is a positive integer. How many grime numbers are there in the sequence $11, 21, 31, 41, \ldots, 981, 991$?

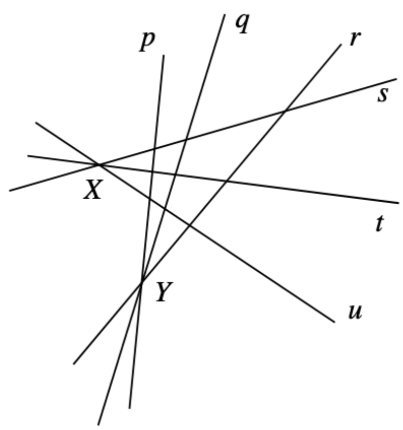
MA68. $PQRS$ is a square. The points T and U are the midpoints of QR and RS respectively. The line QS cuts PT and PU at W and V respectively. What fraction of the area of the square $PQRS$ is the area of the pentagon $RTWVU$?



MA69. The diagram shows two straight lines PR and QS crossing at O. What is the value of x?



MA70. Challengeborough's underground train network consists of six lines, p, q, r, s, t, u, as shown. Wherever two lines meet, there is a station which enables passengers to change lines. On each line, each train stops at every station. Jessica wants to travel from station X to station Y. She does not want to use any line more than once, nor return to station X after leaving it, nor leave station Y having reached it. How many different routes, satisfying these conditions, can she choose?



. . . .

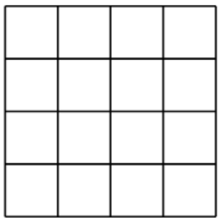
Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juin 2020**.

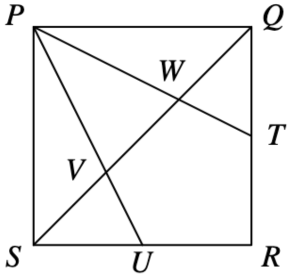
La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA66. Les 16 petits carrés illustrés ci-bas sont tous de côtés 1 unité. Combien de paires de sommets se trouvent à une distance entière d'unités?

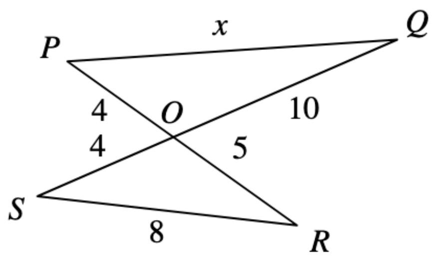


MA67. Soient les entiers de la forme $10n + 1$, où n est entier positif. Un tel nombre est dit *remier* s'il n'est pas possible de le représenter comme produit de deux plus petits entiers possiblement égaux, toujours de la forme $10k + 1$ où k serait entier positif. Combien de nombres remiers y a-t-il parmi $11, 21, 31, 41, \ldots, 981, 991$?

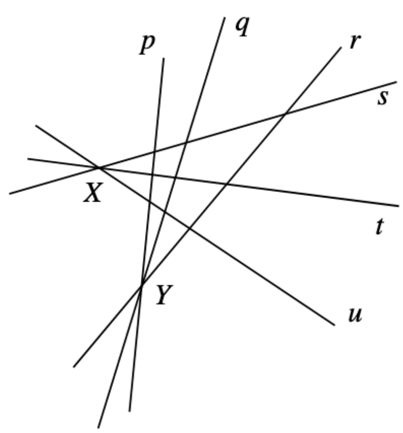
MA68. $PQRS$ est un carré. Les points T et U sont les mi points de QR et RS respectivement. La ligne QS intersecte PT et PU en W et V respectivement. Quelle fraction de la surface du carré $PQRS$ est occupée par le pentagone $RTWVU$?



MA69. Le diagramme ci-bas montre deux lignes PR et QS intersectant en O. Quelle est la valeur de x?



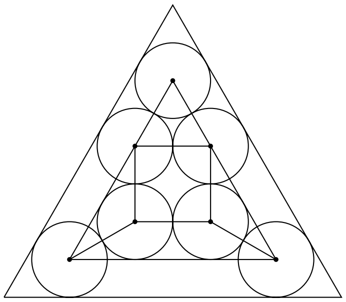
MA70. Le métro de Winnibourg consiste de six lignes, p, q, r, s, t, u, telles qu'indiquées ci-bas. Lorsque deux lignes se rencontrent, on y retrouve une station permettant de changer de ligne. De plus, le métro s'arrête à toute station sur sa ligne. Jéhane désire voyager de la station X à la station Y. Mais elle refuse d'utiliser une quelconque ligne plus qu'une fois, en plus de ne jamais revenir une deuxième fois à la station X, ni de quitter la station Y après y être arrivée. Déterminer le nombre de telles routes différentes.



MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(9), p. 495–496.

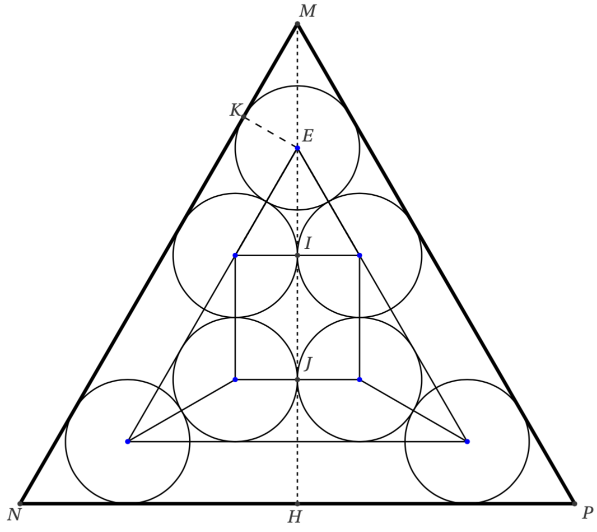
MA41. The diagram shows the densest packing of seven circles in an equilateral triangle.



Determine the exact fraction of the area of the triangle that is covered by the circles.

Originally from "Shaking Hands in Corner Brook and Other Math Problems" by Peter Booth, Bruce Shawyer and John Grant McLoughlin.

We received 7 submissions, of which 6 were correct and complete. We present the solution by Dominique Mouchet, modified by the editor.



On prend 1 comme longueur du côté du grand triangle équilatéral. On notera R le rayon des cercles. Exprimons la hauteur MH en fonction de R:

- le triangle MKE a pour angles $90-60-30$ et $EK = R$. Donc $ME = 2R$.
- EI est la hauteur d'un triangle équilatéral de côté $2R$. Donc

$$EI = 2R \cdot \frac{\sqrt{3}}{2} = R\sqrt{3}.$$

- $IJ = JH = 2R$.

Donc

$$MH = ME + EI + IJ + IH = 2R + R\sqrt{3} + 2R + 2R = R\big(6+\sqrt{3}\big).$$

Comme $MH = \dfrac{\sqrt{3}}{2}$, on obtient

$$R = \frac{\sqrt{3}}{2\big(6+\sqrt{3}\big)} = \frac{\sqrt{3}\big(6-\sqrt{3}\big)}{2\cdot 33} = \frac{2\sqrt{3}-1}{22}.$$

La fraction p de la surface du triangle couverte par les 7 triangles est donc:

$$p = \frac{7\pi R^2}{\frac{\sqrt{3}}{4}} = \frac{28\pi}{\sqrt{3}}\left(\frac{13-4\sqrt{3}}{484}\right) = \frac{7\pi}{363}\left(13\sqrt{3}-12\right) \approx 0.6371.$$

MA42. Find all functions of the form $f(x) = \dfrac{a+bx}{b+x}$ where a and b are constants such that $f(2) = 2f(5)$ and $f(0) + 3f(-2) = 0$.

Originally Question 2 of 1980 J.I.R. McKnight Mathematics Scholarship Paper.

We received 8 submissions, all of which were correct and complete. We present the solution by José Luis Díaz-Barrero, modified by the editor.

The condition gives us that

$$\frac{a+2b}{b+2} = 2\left(\frac{a+5b}{b+5}\right) \quad \text{and} \quad \frac{a}{b} + 3\left(\frac{a-2b}{b-2}\right) = 0.$$

The above results in the nonlinear system of equations:

$$\begin{aligned} 8b^2 + (10+a)b - a &= 0, \\ 6b^2 - 4ab + 2a &= 0. \end{aligned} \tag{1}$$

The resultant of the above system is

$$-4a(a+1)(19a-300).$$

Substituting the zeros of a in the above we see that $(a,b)=(0,0)$, $(a,b)=(-1,-1)$ and $(a,b)=(300/19,10/19)$ solve (1). Thus

$$f(x)=0,\quad f(x)=\frac{1+x}{1-x},\quad \text{and}\quad f(x)=\frac{300+10x}{10+19x},$$

are the only functions which satisfy the stated condition.

MA43. If n is not divisible by 4, prove that $1^n+2^n+3^n+4^n$ is divisible by 5 for any positive integer n.

Adapted from Problem 2 of the 1901 Competition in Hungarian Problem Book 1 (1963).

We received 13 submissions, all of which were correct and complete. We present the generalized solution by the Problem Solving Group from Missouri State University, modified by the editor.

We will show, more generally, that if p is any prime number and n is a positive integer then

$$1^n+2^n+3^n+\ldots+(p-1)^n$$

is a multiple of p if and only if n is not divisible by $p-1$.

It is well known that since p is prime, there is an element $\alpha\in\mathbb{Z}_p$ (a primitive root) such that for all $i\in\mathbb{Z}_p, i=\alpha^k$ for some integer k, with $0\le k\le p-2$. Therefore

$$\sum_{i=1}^{p-1} i^n \equiv \sum_{k=0}^{p-2}\left(\alpha^k\right)^n \equiv \sum_{k=0}^{p-2}\left(\alpha^n\right)^k.$$

Note that $\alpha^n\equiv 1 \bmod p$ if and only if n is a multiple of $p-1$.

We prove both directions:

$\Rightarrow$ (Contrapositive) If n is a multiple of $p-1$, $\alpha^n\equiv 1 \bmod p$ and

$$\sum_{i=1}^{p-1} i^n \equiv \sum_{k=0}^{p-2}(\alpha^n)^k \equiv p-1 \not\equiv 0 \bmod p.$$

Thus our sum is not a multiple of p.

$\Leftarrow$ If n is not a multiple of $p-1$, $\alpha^n-1\not\equiv 0 \bmod p$, so $\alpha^n-1\in\mathbb{Z}_p$. Using the formula for finite geometric series (with $b=\alpha^n$), we have

$$\sum_{i=1}^{p-1} i^n \bmod p \equiv \sum_{k=0}^{p-2}(\alpha^n)^k \bmod p \equiv \left(b^{p-1}-1\right)(b-1)^{-1} \bmod p \equiv 0 \bmod p.$$

Thus our sum is a multiple of p.

We observe the original problem considers the case when $p=5$.

MA44. Find the largest positive integer which divides all expressions of the form $n^5 - n^3$ where n is a positive integer. Justify your answer.

Proposed by John McLoughlin.

We received 11 submissions, all of which were correct and complete. We present the joint solution by the Problem Solving Group from Missouri State University and Tianqi Jiang (solved independently), modified by the editor.

First note that when $n = 2$, we have that $2^5 - 2^3 = 24$. Thus the number we seek must be a factor of 24.

We show $3 \mid n^5 - n^3$. As $n^5 - n^3 = n^3(n+1)(n-1)$ is divisible by three consecutive integers, it follows one of these numbers is a multiple of 3. Thus $3 \mid n^5 - n^3$.

We show $8 \mid n^5 - n^3$ by considering cases. If $n = 2k$, then

$$n^5 - n^3 = (2k)^3(2k+1)(2k-1) = 8k^3(2k+1)(2k-1).$$

Thus $8 \mid n^5 - n^3$. If $n = 2k+1$, then

$$n^5 - n^3 = (2k+1)^3(2k+2)(2k) = (2k+1)^3 \cdot 2^2 \cdot k \cdot (k+1).$$

As one of k or $k+1$ is even, it follows $8 \mid n^5 - n^3$.

Since 3 and 8 are relatively prime, $24 \mid n^5 - n^3$. As we established 24 as an upper bound, our proof is complete.

MA45. A sequence $s_1, s_2, \ldots, s_n$ is harmonic if the reciprocals of the terms are in arithmetic sequence. Suppose $s_1, s_2, \ldots, s_{10}$ are in harmonic sequence. Given $s_1 = 1.2$ and $s_{10} = 3.68$, find $s_1 + s_2 + \cdots + s_{10}$.

Originally Question 11 of 1988 Illinois CTM, State Finals AA.

We received 2 submissions, both correct and complete. We present the solution by Doddy Kastanya.

The arithmetic sequence of interest is $a_1, a_2, \ldots, a_{10}$ where

$$s_1 = \frac{1}{a_1},\ s_2 = \frac{1}{a_2}, \ldots, s_{10} = \frac{1}{a_{10}}.$$

From the problem statement, we know that $a_1 = \frac{1}{1.2} = \frac{5}{6}$ and $a_{10} = \frac{1}{3.68} = \frac{100}{368}$.

For the arithmetic sequence, there are eight items in between a_1 and a_{10} with equal spacing. Turning the denominator for these two values to 9936, we get $a_1 = \frac{8280}{9936}$ and $a_{10} = \frac{2700}{9936}$. The spacing between two numbers is $\frac{620}{9936}$. With this knowledge, the other items can be determined: $a_2 = \frac{7660}{9936}$, $a_3 = \frac{7040}{9936}$, up to $a_9 = \frac{3320}{9936}$.

The corresponding values of s_1 through s_{10} can be easily calculated. Finally, the sum of s_1 through s_{10} is calculated as 20.46.

PROBLEM SOLVING VIGNETTES

No. 11

Shawn Godin

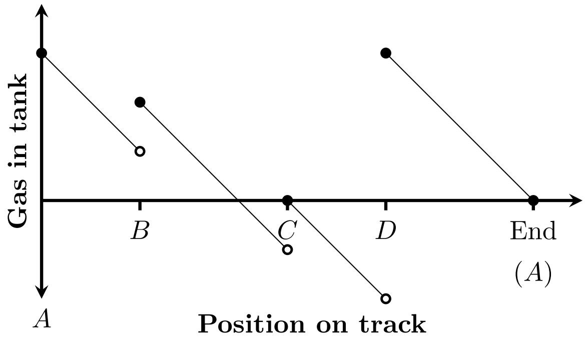
Picking Representations

In many cases, a problem's solution is aided by thinking about the problem in a different way than it was originally presented. This may be by looking at a different, but related problem whose solution leads back to the original. We can also think about a problem differently by choosing some other way to represent it. Analytic geometry is an example, where we can think of geometric problems algebraically or algebraic problems geometrically.

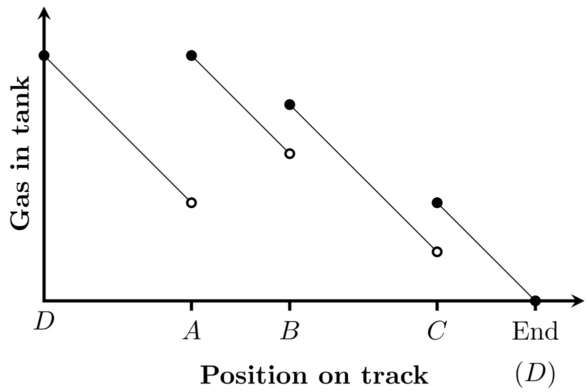
When I thought about this topic the following problem came to mind:

> *A gas powered go-cart is empty and on a track. Around the track are a number of gas cans. The total amount of gas in all the cans is equal to the amount of gas needed to go around the track once. Show that, no matter how the gas and cans are distributed, you can find a place to start so that you can make it all the way around the track.*

I was introduced to this problem by *Crux* Editorial Board member Ed Barbeau at a workshop he did for teachers over 20 years ago. It's one of those problems you can convince yourself must work, but coming up with an airtight argument that convinces others is another thing. The key to the insightful solution that was given by Ed was to imagine that we are allowed to have a "negative" amount of gas in our tank. Then the graph of the gas in our tank versus the distance driven will be a piecewise linear function where all the pieces of the graph will have equal, negative slopes; there will be a step discontinuity at the location of each gas can; and when we have finished one trip around the track our tank will, again, be empty.



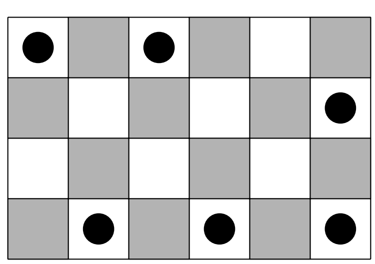
Thus if we draw the graph and find the lowest point, this will be the place that we should start.



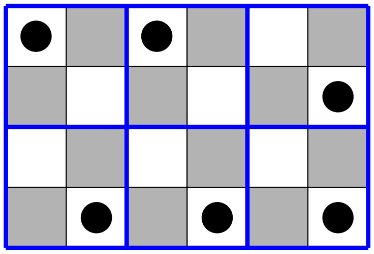
For example, in the graphs above, we are assuming that there are four gas cans A, B, C, and D. If we start at can A, we get the first graph above on the previous page. Thus, we see that we should have started at can D, which would have given us the second graph above.

Choosing the graphical representation not only helped make our argument clearer, it also gave way to the solution. Now, let's consider Problem 3 from the 2019 Canadian Mathematical Olympiad:

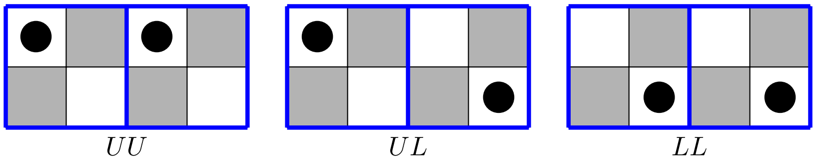
> *Let m and n be positive integers. A $2m \times 2n$ grid of squares is coloured in the usual chessboard fashion. Find the number of ways of placing mn counters on the white squares, at most one counter per square, so that no two counters are on white squares that are diagonally adjacent. An example of a way to place the counters when $m = 2$ and $n = 3$ is shown below.*



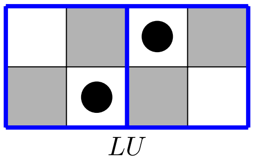
Since counters cannot be on diagonally adjacent squares, any 2×2 square drawn on the grid can only contain at most one counter. This suggests partitioning the grid into 2×2 squares, as in the diagram below. Since there are $\frac{(2m)(2n)}{(2)(2)} = mn$ 2×2 squares on the grid, each of these squares will have exactly one counter.



Notice that there are two configurations that the 2×2 squares can be in: either the counter can be in the upper corner (U), or the lower corner (L). When we look at two 2×2 squares beside each other we see that UU, UL and LL are all valid configurations.



On the other hand the configuration LU is not allowed.



Similarly, going from top to bottom we can have U followed by U or L, but L can only be followed by another L. Hence, our original $2m \times 2n$ grid was replaced with an $m \times n$ grid of 2×2 squares, which in turn can be replaced by an $m \times n$ grid filled with the symbols U and L.

The original example can now be replaced with

U	U	L
L	L	L

Notice that in any row or column we have a number of Us (possibly none) followed by a number of Ls. Once an L appears in a row or column, all entries to the right and below it are also Ls. Thus starting from the top row and proceeding downward, each new row has at least as many Ls as the row above it, and possibly more.

Considering the setup from the example in the problem statement, in which a 4×6 starting grid reduces to a 2×3 grid of Us and Ls, we can readily construct the 10 ways that the grid can be filled and count the number of Ls in each row (the case in bold blue is the example from the problem statement).

U	U	U	0
U	U	U	0

U	U	U	0
U	U	L	1

U	U	U	0
U	L	L	2

U	U	U	0
L	L	L	3

U	U	L	1
U	U	L	1

U	U	L	1
U	L	L	2

U	U	L	1
L	L	L	3

U	L	L	2
U	L	L	2

U	L	L	2
L	L	L	3

L	L	L	3
L	L	L	3

Thus we can think of our problem in another way: a sequence of two non-negative integers representing the number of Ls in each row. For the configuration considered above these would be:

$$0,0;\quad 0,1;\quad 0,2;\quad 0,3;\quad 1,1;\quad 1,2;\quad 1,3;\quad 2,2;\quad 2,3;\quad 3,3.$$

Notice that, to satisfy the conditions of the problem, all of these sequences are non-decreasing, containing numbers less than or equal to 3 (the number of columns). Thus our problem is equivalent to finding the number of non-decreasing sequences of m terms chosen (possibly with repetition) from the set $\{0,1,2,\ldots,n\}$. In the example that would be the number of non-decreasing sequences of 2 terms from the set $\{0,1,2,3\}$.

We will choose another representation to attack the sequence problem. Since my numbers can be as large as 3, we will use 3 stars. Since we have 2 terms, we will use 2 bars. We will arrange these 5 symbols in some order, like

$$*|**|$$

This arrangement is interpreted by counting all the stars to the left of the leftmost bar, 1, and all the stars to the left of the rightmost bar, 3. Hence the arrangement above represents the non-decreasing sequence of 2 terms from the set $\{0,1,2,3\}$: $1,3$. Each sequence is represented by a unique arrangement of stars and bars, and each arrangement of stars and bars corresponds to a unique sequence. Hence counting the number of arrangements of 3 stars and 2 bars gives us the solution to the configuration in the problem statement.

We can count the number of ways to arrange the 3 stars and 2 bars in several ways to get $\dfrac{5!}{3!2!} = 10$. This can be interpreted as arranging 5 things, 3 of one type (stars) and 2 of another (bars). We can also interpret this as we have 5 positions to put our symbols and we must choose 3 of them to put the stars, leaving the rest of the places for bars. Alternatively, we could have picked the places for the bars first yielding $\binom{5}{2} = \binom{5}{3} = 10$.

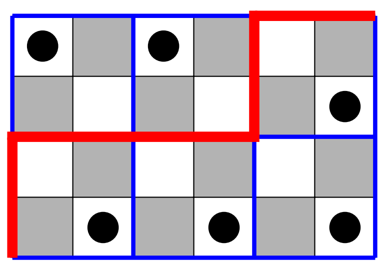
Returning to the problem, in the general case we have a $2m \times 2n$ grid of squares filled with counters. We are representing this by a non-decreasing sequence of m numbers, where each number is less than or equal to n, which represents the number of L's in our smaller grid. Converting this to m bars and n stars, we get the total number of ways our task can be completed is

$$\binom{m+n}{m} = \binom{m+n}{n} = \frac{(m+n)!}{m!n!}.$$

It is worthwhile to go back to the problem and convince yourself that you would get the same result with these slight variations on our technique:

- Counting the number of non-increasing sequences representing the number of Us in each row.
- Counting the number of non-increasing sequences representing the number of Us in each column.
- Counting the number of non-decreasing sequences representing the number of Ls in each column.

The official solution to the problem uses a slightly different approach. It also notices the differences between the Us and Ls and notes that the boundary separating these two types of 2×2 cells makes a path from the lower left corner of the big grid to the upper right corner, travelling either to the right or up. The original example with the boundary highlighted in red is in the diagram below. As an exercise, you may want to solve the problem using this representation.



Keep in mind that sometimes changing your point of view through a different representation of the problem may lead you to a solution. You may want to check out the seventh number of this column, *Counting Carefully* [2019: 386-389], where the stars and bars technique was used in a slightly different way. For your enjoyment, here are the rest of the problems from the 2019 Canadian Mathematical Olympiad.

The 2019 Canadian Mathematical Olympiad

1. Amy has drawn three points in a plane, A, B, and C, such that $AB = BC = CA = 6$. Amy is allowed to draw a new point if it is the circumcenter of a triangle whose vertices she has already drawn. For example, she can draw the circumcenter O of triangle ABC, and then afterwards she can draw the circumcenter of triangle ABO.

(a) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 7.

(b) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 2019.

(Recall that the circumcenter of a triangle is the center of the circle that passes through its three vertices.)

2. Let a and b be positive integers such that $a+b^3$ is divisible by $a^2+3ab+3b^2-1$. Prove that $a^2+3ab+3b^2-1$ is divisible by the cube of an integer greater than 1.

4. Let n be an integer greater than 1, and let $a_0, a_1, \ldots, a_n$ be real numbers with $a_1 = a_{n-1} = 0$. Prove that for any real number k,

$$|a_0| - |a_n| \leq \sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}|.$$

5. David and Jacob are playing a game of connecting $n \geq 3$ points drawn in a plane. No three of the points are collinear. On each player's turn, he chooses two points to connect by a new line segment. The first player to complete a cycle consisting of an odd number of line segments loses the game. (Both endpoints of each line segment in the cycle must be among the n given points, not points which arise later as intersections of segments.) Assuming David goes first, determine all n for which he has a winning strategy.

The author would like to thank Ed Barbeau for reminding him about the details of the go-cart problem and providing valuable feedback that greatly improved the article.