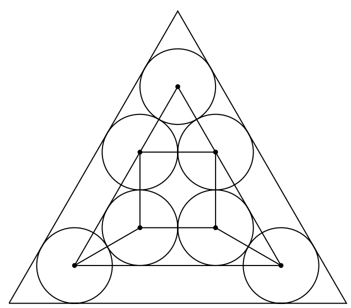
MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(9), p. 495–496.

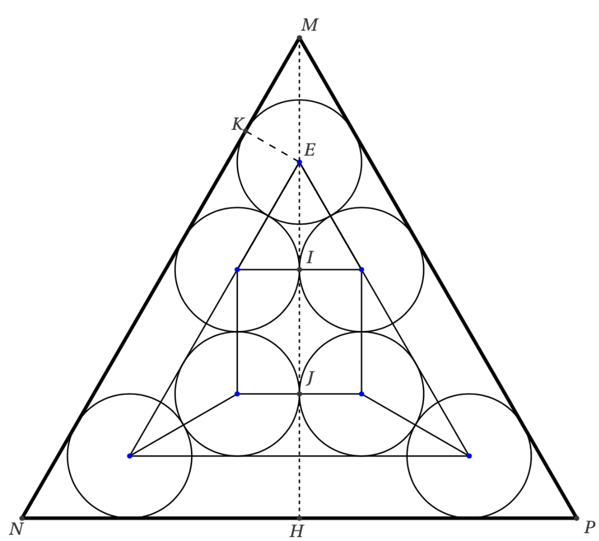
MA41. The diagram shows the densest packing of seven circles in an equilateral triangle.



Determine the exact fraction of the area of the triangle that is covered by the circles.

Originally from "Shaking Hands in Corner Brook and Other Math Problems" by Peter Booth, Bruce Shawyer and John Grant McLoughlin.

We received 7 submissions, of which 6 were correct and complete. We present the solution by Dominique Mouchet, modified by the editor.



On prend 1 comme longueur du côté du grand triangle équilatéral. On notera R le rayon des cercles. Exprimons la hauteur MH en fonction de R:

- le triangle MKE a pour angles $90 - 60 - 30$ et $EK = R$. Donc $ME = 2R$.
- EI est la hauteur d'un triangle équilatéral de côté $2R$. Donc
$$EI = 2R \cdot \frac{\sqrt{3}}{2} = R\sqrt{3}.$$
- $IJ = JH = 2R$.

Donc
$$MH = ME + EI + IJ + IH = 2R + R\sqrt{3} + 2R + 2R = R\big(6 + \sqrt{3}\big).$$

Comme $MH = \dfrac{\sqrt{3}}{2}$, on obtient
$$R = \frac{\sqrt{3}}{2\big(6+\sqrt{3}\big)} = \frac{\sqrt{3}\big(6-\sqrt{3}\big)}{2 \cdot 33} = \frac{2\sqrt{3}-1}{22}.$$

La fraction p de la surface du triangle couverte par les 7 triangles est donc:
$$p = \frac{7\pi R^2}{\frac{\sqrt{3}}{4}} = \frac{28\pi}{\sqrt{3}}\left(\frac{13-4\sqrt{3}}{484}\right) = \frac{7\pi}{363}\left(13\sqrt{3}-12\right) \approx 0.6371.$$

MA42. Find all functions of the form $f(x) = \dfrac{a+bx}{b+x}$ where a and b are constants such that $f(2) = 2f(5)$ and $f(0) + 3f(-2) = 0$.

Originally Question 2 of 1980 J.I.R. McKnight Mathematics Scholarship Paper.

We received 8 submissions, all of which were correct and complete. We present the solution by José Luis Díaz-Barrero, modified by the editor.

The condition gives us that
$$\frac{a+2b}{b+2} = 2\left(\frac{a+5b}{b+5}\right) \quad \text{and} \quad \frac{a}{b} + 3\left(\frac{a-2b}{b-2}\right) = 0.$$

The above results in the nonlinear system of equations:
$$\begin{aligned} 8b^2 + (10+a)b - a &= 0, \\ 6b^2 - 4ab + 2a &= 0. \end{aligned} \tag{1}$$

The resultant of the above system is
$$-4a(a+1)(19a-300).$$

Substituting the zeros of a in the above we see that $(a, b) = (0, 0)$, $(a, b) = (-1, -1)$ and $(a, b) = (300/19, 10/19)$ solve (1). Thus

$$f(x) = 0, \quad f(x) = \frac{1+x}{1-x}, \quad \text{and} \quad f(x) = \frac{300+10x}{10+19x},$$

are the only functions which satisfy the stated condition.

MA43. If n is not divisible by 4, prove that $1^n + 2^n + 3^n + 4^n$ is divisible by 5 for any positive integer n.

Adapted from Problem 2 of the 1901 Competition in Hungarian Problem Book 1 (1963).

We received 13 submissions, all of which were correct and complete. We present the generalized solution by the Problem Solving Group from Missouri State University, modified by the editor.

We will show, more generally, that if p is any prime number and n is a positive integer then

$$1^n + 2^n + 3^n + \ldots + (p-1)^n$$

is a multiple of p if and only if n is not divisible by $p-1$.

It is well known that since p is prime, there is an element $\alpha \in \mathbb{Z}_p$ (a primitive root) such that for all $i \in \mathbb{Z}_p, i = \alpha^k$ for some integer k, with $0 \leq k \leq p-2$. Therefore

$$\sum_{i=1}^{p-1} i^n \equiv \sum_{k=0}^{p-2} \left(\alpha^k\right)^n \equiv \sum_{k=0}^{p-2} (\alpha^n)^k .$$

Note that $\alpha^n \equiv 1 \bmod p$ if and only if n is a multiple of $p-1$.

We prove both directions:

$\Rightarrow$ (Contrapositive) If n is a multiple of $p-1$, $\alpha^n \equiv 1 \bmod p$ and

$$\sum_{i=1}^{p-1} i^n \equiv \sum_{k=0}^{p-2} (\alpha^n)^k \equiv p - 1 \not\equiv 0 \bmod p.$$

Thus our sum is not a multiple of p.

$\Leftarrow$ If n is not a multiple of $p-1$, $\alpha^n - 1 \not\equiv 0 \bmod p$, so $\alpha^n - 1 \in \mathbb{Z}_p$. Using the formula for finite geometric series (with $b = \alpha^n$), we have

$$\sum_{i=1}^{p-1} i^n \bmod p \equiv \sum_{k=0}^{p-2} (\alpha^n)^k \bmod p \equiv \left(b^{p-1} - 1\right)(b-1)^{-1} \bmod p \equiv 0 \bmod p.$$

Thus our sum is a multiple of p.

We observe the original problem considers the case when $p = 5$.

MA44. Find the largest positive integer which divides all expressions of the form $n^5 - n^3$ where n is a positive integer. Justify your answer.

Proposed by John McLoughlin.

We received 11 submissions, all of which were correct and complete. We present the joint solution by the Problem Solving Group from Missouri State University and Tianqi Jiang (solved independently), modified by the editor.

First note that when $n = 2$, we have that $2^5 - 2^3 = 24$. Thus the number we seek must be a factor of 24.

We show $3 \mid n^5 - n^3$. As $n^5 - n^3 = n^3(n+1)(n-1)$ is divisible by three consecutive integers, it follows one of these numbers is a multiple of 3. Thus $3 \mid n^5 - n^3$.

We show $8 \mid n^5 - n^3$ by considering cases. If $n = 2k$, then

$$n^5 - n^3 = (2k)^3(2k+1)(2k-1) = 8k^3(2k+1)(2k-1).$$

Thus $8 \mid n^5 - n^3$. If $n = 2k+1$, then

$$n^5 - n^3 = (2k+1)^3(2k+2)(2k) = (2k+1)^3 \cdot 2^2 \cdot k \cdot (k+1).$$

As one of k or $k+1$ is even, it follows $8 \mid n^5 - n^3$.

Since 3 and 8 are relatively prime, $24 \mid n^5 - n^3$. As we established 24 as an upper bound, our proof is complete.

MA45. A sequence $s_1, s_2, \ldots, s_n$ is harmonic if the reciprocals of the terms are in arithmetic sequence. Suppose $s_1, s_2, \ldots, s_{10}$ are in harmonic sequence. Given $s_1 = 1.2$ and $s_{10} = 3.68$, find $s_1 + s_2 + \cdots + s_{10}$.

Originally Question 11 of 1988 Illinois CTM, State Finals AA.

We received 2 submissions, both correct and complete. We present the solution by Doddy Kastanya.

The arithmetic sequence of interest is $a_1, a_2, \ldots, a_{10}$ where

$$s_1 = \frac{1}{a_1},\ s_2 = \frac{1}{a_2}, \ldots, s_{10} = \frac{1}{a_{10}}.$$

From the problem statement, we know that $a_1 = \frac{1}{1.2} = \frac{5}{6}$ and $a_{10} = \frac{1}{3.68} = \frac{100}{368}$.

For the arithmetic sequence, there are eight items in between a_1 and a_{10} with equal spacing. Turning the denominator for these two values to 9936, we get $a_1 = \frac{8280}{9936}$ and $a_{10} = \frac{2700}{9936}$. The spacing between two numbers is $\frac{620}{9936}$. With this knowledge, the other items can be determined: $a_2 = \frac{7660}{9936}$, $a_3 = \frac{7040}{9936}$, up to $a_9 = \frac{3320}{9936}$.

The corresponding values of s_1 through s_{10} can be easily calculated. Finally, the sum of s_1 through s_{10} is calculated as 20.46.