PROBLEM SOLVING VIGNETTES

No. 11

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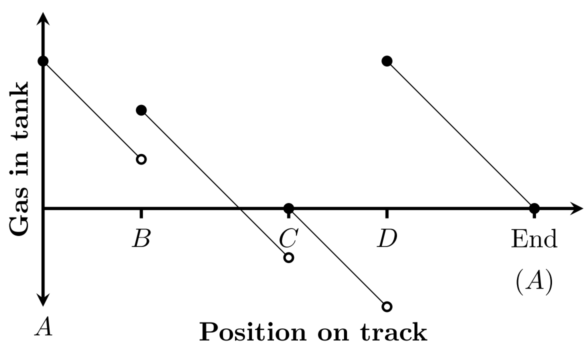
Picking Representations

In many cases, a problem's solution is aided by thinking about the problem in a different way than it was originally presented. This may be by looking at a different, but related problem whose solution leads back to the original. We can also think about a problem differently by choosing some other way to represent it. Analytic geometry is an example, where we can think of geometric problems algebraically or algebraic problems geometrically.

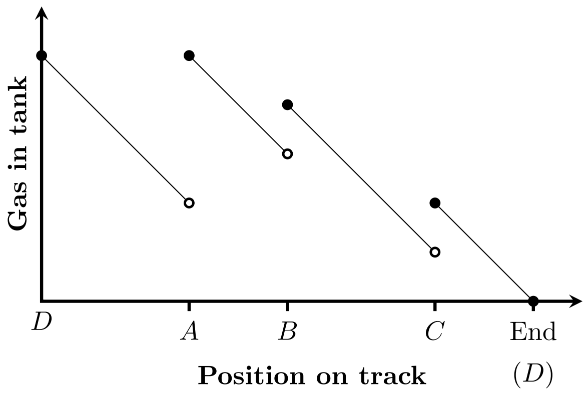
When I thought about this topic the following problem came to mind:

> *A gas powered go-cart is empty and on a track. Around the track are a number of gas cans. The total amount of gas in all the cans is equal to the amount of gas needed to go around the track once. Show that, no matter how the gas and cans are distributed, you can find a place to start so that you can make it all the way around the track.*

I was introduced to this problem by *Crux* Editorial Board member Ed Barbeau at a workshop he did for teachers over 20 years ago. It's one of those problems you can convince yourself must work, but coming up with an airtight argument that convinces others is another thing. The key to the insightful solution that was given by Ed was to imagine that we are allowed to have a "negative" amount of gas in our tank. Then the graph of the gas in our tank versus the distance driven will be a piecewise linear function where all the pieces of the graph will have equal, negative slopes; there will be a step discontinuity at the location of each gas can; and when we have finished one trip around the track our tank will, again, be empty.



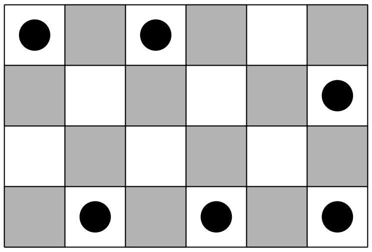
Thus if we draw the graph and find the lowest point, this will be the place that we should start.



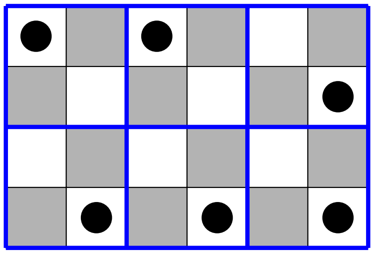
For example, in the graphs above, we are assuming that there are four gas cans A, B, C, and D. If we start at can A, we get the first graph above on the previous page. Thus, we see that we should have started at can D, which would have given us the second graph above.

Choosing the graphical representation not only helped make our argument clearer, it also gave way to the solution. Now, let's consider Problem 3 from the 2019 Canadian Mathematical Olympiad:

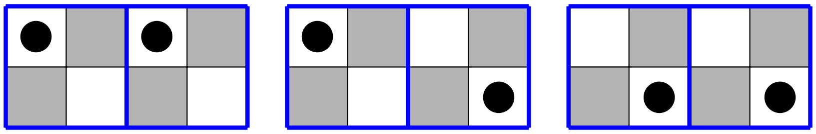
> *Let m and n be positive integers. A $2m \times 2n$ grid of squares is coloured in the usual chessboard fashion. Find the number of ways of placing mn counters on the white squares, at most one counter per square, so that no two counters are on white squares that are diagonally adjacent. An example of a way to place the counters when $m = 2$ and $n = 3$ is shown below.*



Since counters cannot be on diagonally adjacent squares, any 2×2 square drawn on the grid can only contain at most one counter. This suggests partitioning the grid into 2×2 squares, as in the diagram below. Since there are $\frac{(2m)(2n)}{(2)(2)} = mn$ 2×2 squares on the grid, each of these squares will have exactly one counter.

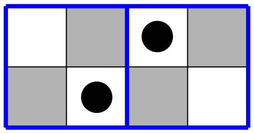


Notice that there are two configurations that the 2×2 squares can be in: either the counter can be in the upper corner (U), or the lower corner (L). When we look at two 2×2 squares beside each other we see that UU, UL and LL are all valid configurations.



UU UL LL

On the other hand the configuration LU is not allowed.



LU

Similarly, going from top to bottom we can have U followed by U or L, but L can only be followed by another L. Hence, our original $2m \times 2n$ grid was replaced with an $m \times n$ grid of 2×2 squares, which in turn can be replaced by an $m \times n$ grid filled with the symbols U and L.

The original example can now be replaced with

U	U	L
L	L	L

Notice that in any row or column we have a number of Us (possibly none) followed by a number of Ls. Once an L appears in a row or column, all entries to the right and below it are also Ls. Thus starting from the top row and proceeding downward, each new row has at least as many Ls as the row above it, and possibly more.

Considering the setup from the example in the problem statement, in which a 4×6 starting grid reduces to a 2×3 grid of Us and Ls, we can readily construct the 10 ways that the grid can be filled and count the number of Ls in each row (the case in bold blue is the example from the problem statement).

U	U	U	0
U	U	U	0

U	U	U	0
U	U	L	1

U	U	U	0
U	L	L	2

U	U	U	0
L	L	L	3

U	U	L	1
U	U	L	1

U	U	L	1
U	L	L	2

U	U	L	1
L	L	L	3

U	L	L	2
U	L	L	2

U	L	L	2
L	L	L	3

L	L	L	3
L	L	L	3

Thus we can think of our problem in another way: a sequence of two non-negative integers representing the number of Ls in each row. For the configuration considered above these would be:

$$0,0;\quad 0,1;\quad 0,2;\quad 0,3;\quad 1,1;\quad 1,2;\quad 1,3;\quad 2,2;\quad 2,3;\quad 3,3.$$

Notice that, to satisfy the conditions of the problem, all of these sequences are non-decreasing, containing numbers less than or equal to 3 (the number of columns). Thus our problem is equivalent to finding the number of non-decreasing sequences of m terms chosen (possibly with repetition) from the set $\{0,1,2,\ldots,n\}$. In the example that would be the number of non-decreasing sequences of 2 terms from the set $\{0,1,2,3\}$.

We will choose another representation to attack the sequence problem. Since my numbers can be as large as 3, we will use 3 stars. Since we have 2 terms, we will use 2 bars. We will arrange these 5 symbols in some order, like

$$*|**|$$

This arrangement is interpreted by counting all the stars to the left of the leftmost bar, 1, and all the stars to the left of the rightmost bar, 3. Hence the arrangement above represents the non-decreasing sequence of 2 terms from the set $\{0,1,2,3\}$: $1,3$. Each sequence is represented by a unique arrangement of stars and bars, and each arrangement of stars and bars corresponds to a unique sequence. Hence counting the number of arrangements of 3 stars and 2 bars gives us the solution to the configuration in the problem statement.

We can count the number of ways to arrange the 3 stars and 2 bars in several ways to get $\dfrac{5!}{3!2!} = 10$. This can be interpreted as arranging 5 things, 3 of one type (stars) and 2 of another (bars). We can also interpret this as we have 5 positions to put our symbols and we must choose 3 of them to put the stars, leaving the rest of the places for bars. Alternatively, we could have picked the places for the bars first yielding $\binom{5}{2} = \binom{5}{3} = 10$.

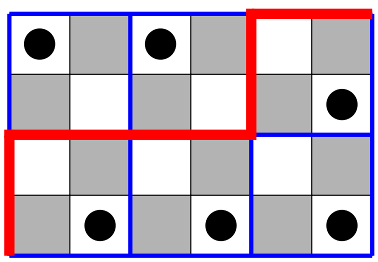
Returning to the problem, in the general case we have a $2m \times 2n$ grid of squares filled with counters. We are representing this by a non-decreasing sequence of m numbers, where each number is less than or equal to n, which represents the number of L's in our smaller grid. Converting this to m bars and n stars, we get the total number of ways our task can be completed is

$$\binom{m+n}{m} = \binom{m+n}{n} = \frac{(m+n)!}{m!n!}.$$

It is worthwhile to go back to the problem and convince yourself that you would get the same result with these slight variations on our technique:

- Counting the number of non-increasing sequences representing the number of Us in each row.
- Counting the number of non-increasing sequences representing the number of Us in each column.
- Counting the number of non-decreasing sequences representing the number of Ls in each column.

The official solution to the problem uses a slightly different approach. It also notices the differences between the Us and Ls and notes that the boundary separating these two types of 2×2 cells makes a path from the lower left corner of the big grid to the upper right corner, travelling either to the right or up. The original example with the boundary highlighted in red is in the diagram below. As an exercise, you may want to solve the problem using this representation.



Keep in mind that sometimes changing your point of view through a different representation of the problem may lead you to a solution. You may want to check out the seventh number of this column, *Counting Carefully* [2019: 386-389], where the stars and bars technique was used in a slightly different way. For your enjoyment, here are the rest of the problems from the 2019 Canadian Mathematical Olympiad.

The 2019 Canadian Mathematical Olympiad

1. Amy has drawn three points in a plane, A, B, and C, such that $AB = BC = CA = 6$. Amy is allowed to draw a new point if it is the circumcenter of a triangle whose vertices she has already drawn. For example, she can draw the circumcenter O of triangle ABC, and then afterwards she can draw the circumcenter of triangle ABO.

(a) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 7.

(b) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 2019.

(Recall that the circumcenter of a triangle is the center of the circle that passes through its three vertices.)

2. Let a and b be positive integers such that $a+b^3$ is divisible by $a^2+3ab+3b^2-1$. Prove that $a^2+3ab+3b^2-1$ is divisible by the cube of an integer greater than 1.

4. Let n be an integer greater than 1, and let $a_0, a_1, \ldots, a_n$ be real numbers with $a_1 = a_{n-1} = 0$. Prove that for any real number k,

$$|a_0| - |a_n| \leq \sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}|.$$

5. David and Jacob are playing a game of connecting $n \geq 3$ points drawn in a plane. No three of the points are collinear. On each player's turn, he chooses two points to connect by a new line segment. The first player to complete a cycle consisting of an odd number of line segments loses the game. (Both endpoints of each line segment in the cycle must be among the n given points, not points which arise later as intersections of segments.) Assuming David goes first, determine all n for which he has a winning strategy.

The author would like to thank Ed Barbeau for reminding him about the details of the go-cart problem and providing valuable feedback that greatly improved the article.