BONUS PROBLEMS

*These problems appear as a bonus. Their solutions will **not** be considered for publication.*

B1. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let $ABCD$ be a cyclic quadrilateral. Let P be a point on the arc BC and let L and Q be the feet of perpendiculars dropped from P on the sides AD and BC, respectively. Let M and N be the feet of perpendiculars dropped from P on the lines AB and DC, respectively. Prove that

$$\frac{AM}{MB} \cdot \frac{BQ}{QC} \cdot \frac{CN}{ND} \cdot \frac{DL}{LA} = 1.$$

B2. *Proposed by Leonard Giugiuc.*

Find the real numbers x, y, z and t such that

$$xt - yz = -1 \quad \text{and} \quad x^2 + y^2 + z^2 + t^2 - xz - yt = \sqrt{3}.$$

B3. *Proposed by Leonard Giugiuc.*

Let a, b and c be positive real numbers such that $ab + bc + ca = 3$. Prove the inequality

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \leq 1.$$

B4. *Proposed by Leonard Giugiuc.*

Let ABC be a triangle with no angle exceeding 120° with $BC = a, AC = b$ and $AB = c$. Let T be its Fermat-Torricelli point, that is the point such that the total distance from the three vertices of ABC to T is minimum possible. Prove that

$$(b+c)|TA| + (c+a)|TB| + (a+b)|TC| \geq \sqrt{3}(|TA| + |TB| + |TC|)^2 - 4\text{Area}(ABC).$$

B5. *Proposed by Leonard Giugiuc.*

Let n be an integer such that $n \geq 4$. Consider real numbers $a_k, 1 \leq k \leq n$ such that $2 \geq a_1 \geq 1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n$ and $\sum_{k=1}^n a_k = n$. Prove that

a) $\sum_{k=1}^n a_k^2 \leq n + 2$.

b) $\sum_{1 \leq i < j \leq n}^n a_i a_j \geq \frac{(n-2)(n+1)}{2}$.

B6. *Proposed by Leonard Giugiuc.*

Let ABC be a triangle such that $\angle BAC \geq \frac{2\pi}{3}$. Prove that

$$\frac{r}{R} \leq \frac{2\sqrt{3}-3}{2},$$

where r is the inradius and R is the circumradius of ABC.

B7. *Proposed by Leonard Giugiuc.*

Let a, b, c and d be real numbers such that $2 \geq a \geq 1 \geq b \geq c \geq d \geq 0$ and $a+b+c+d=4$. Prove that

$$\frac{2}{a^3+b^3+c^3+d^3} + \frac{9}{ab+bc+cd+da+ac+bd} \leq 2.$$

B8. *Proposed by Leonard Giugiuc and Dan Stefan Marinescu.*

Let $ABCD$ be a rectangle with center O. Let M and P be two points in the plane (not necessarily distinct) such that O lies on the line MP and $OM = 3 \cdot OP$. Prove that

$$MA + MB + MC + MD \geq PA + PB + PC + PD.$$

B9. *Proposed by Leonard Giugiuc.*

Let $a, b, c \geq 1$ and $0 \leq d, e, f \leq 1$ such that $a+b+c+d+e+f=6$. Prove that

$$6 \leq a^2+b^2+c^2+d^2+e^2+f^2 \leq 18.$$

B10. *Proposed by Leonard Giugiuc.*

Let ABC be a nonobtuse triangle with smallest angle A. Prove that

$$\cos(B-C) \geq \cos B + \cos C$$

and determine when equality holds.

B11. *Proposed by Michael Rozenberg and Leonard Giugiuc.*

Prove that if a, b, c and d are non-negative real numbers such that $a+b+c+d=4$, then

$$ab+bc+cd+da+ac+bd \geq 3\sqrt{(a^2+b^2+c^2+d^2)abcd}.$$

B12. *Proposed by Leonard Giugiuc.*

Let x, y and z be positive real numbers such that $xyz = 512$. Prove that

$$\frac{1}{\sqrt{1+x}} + \frac{1}{\sqrt{1+y}} + \frac{1}{\sqrt{1+z}} \geq 1.$$

B13★. *Proposed by Leonard Giugiuc.*

Let n be an integer with $n \geq 4$. Prove or disprove that for any positive real numbers a_i, $i = 1, 2, \ldots, n$ that sum up to 1, we have:

$$\sqrt[n]{(1 - a_1^n)(1 - a_2^n) \cdots (1 - a_n^n)} \geq (n^n - 1)a_1 a_2 \ldots a_n.$$

B14★. *Proposed by Leonard Giugiuc.*

Let k be a real number with $k > \frac{7+3\sqrt{5}}{2}$. Prove or disprove that for any non-negative real numbers x, y, z no two of which are zero, we have

$$\sqrt{\frac{x}{ky+z}} + \sqrt{\frac{y}{kz+x}} + \sqrt{\frac{z}{kx+y}} \geq \frac{3}{\sqrt{k+1}}.$$

B15. *Proposed by Leonard Giugiuc.*

Let a, b and c be real numbers such that $a \geq b \geq 1 \geq c \geq 0$ and $a + b + c = 3$.

a) Show that $2 \leq ab + bc + ca \leq 3$.

b) Prove that $a^3 + b^3 + c^3 + \frac{45}{a^2+b^2+c^2} \leq 18$ and study the equality cases.

B16. *Proposed by Dao Thanh Oai and Leonard Giugiuc.*

Let $ABCD$ be a cyclic quadrilateral. Prove that the following two statements are equivalent:

a) $AC \geq BD$,

b) $AB \cdot AD + CB \cdot CD \geq BA \cdot BC + DA \cdot DC$.

B17. *Proposed by Dao Thanh Oai and Leonard Giugiuc.*

Let $ABCD$ be a cyclic quadrilateral. Prove that

$$AB + AC + AD + BC + BD + CD \leq 4R(\sqrt{2} + 1),$$

where R is the circumradius of $ABCD$.

B18. *Proposed by Leonard Giugiuc and Dorin Marghidanu.*

Let $n \geq 2$ be a natural number, and a_k be real numbers such that $0 < a_k < 2$ for all $k = 1, 2, \ldots, n$ with $\prod_{k=1}^{n} a_k = 1$. Prove that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{1 + a_k}} \leq \frac{n}{\sqrt{2}}.$$

Prove further that the condition $a_k < 2$ can be dropped when $n = 2$ or $n = 3$.

B19. *Proposed by Leonard Giugiuc.*

Find the maximum value k such that

$$a^2 + b^2 + c^2 + k(ab + bc + ca) \geq 3 + k(a + b + c)$$

for any positive numbers a, b and c such that $abc = 1$.

B20$\star$. *Proposed by Leonard Giugiuc.*

Let $x, y \in (0, 3/2)$ be real numbers that satisfy $(x - 2)(y - 2) = 1$. Prove or disprove that

$$x^3 + y^3 \geq 2.$$

B21. *Proposed by Marian Cucoanes and Leonard Giugiuc.*

Consider an arbitrary triangle ABC with medians m_a, m_b, m_c, circumradius R, inradius r and exradii r_a, r_b, r_c. Show that

$$m_a + m_b + m_c \leq \sqrt{16R^2 + 4rR + 9r^2} \leq r_a + r_b + r_c.$$

B22. *Proposed by Leonard Giugiuc.*

Let a, b, c, d, e, f be non-negative real numbers such that $a+b+c+d+e+f = 4$. If $a \geq b \geq c \geq 1 \geq d \geq e \geq f \geq 0$, prove that

$$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + 180abcdef \leq 10.$$

B23. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Given a triangle ABC, let the tangent to its circumcircle at A intersect the line BC at D, and let the circle through A that is tangent to BC at D intersect the circumcircle again at E. Prove that $\dfrac{EB}{EC} = \left(\dfrac{AB}{AC}\right)^3$.

B24. *Proposed by Ruben Dario Auqui and Leonard Giugiuc.*

Let $ABCD$ be a square. Let ω be the circle centered at A with radius AB. A point M lies inside the square on ω; the line BM intersects the side CD at N. Prove that $CM = 2MN$ if and only if CM and BN are perpendicular.

B25. *Proposed by Ruben Dario Auqui and Leonard Giugiuc.*

Let ABC be a triangle with semiperimeter s. The A-excircle of the triangle touches the side BC at Q and the lines AB and AC at M and N, respectively. Suppose that AQ intersects MN at P. Prove that

$$AP = \frac{s\sqrt{a(s-a)(as + (b-c)^2)}}{b(s-c) + c(s-b)}.$$