SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2019: 45(8), p. 476–479.

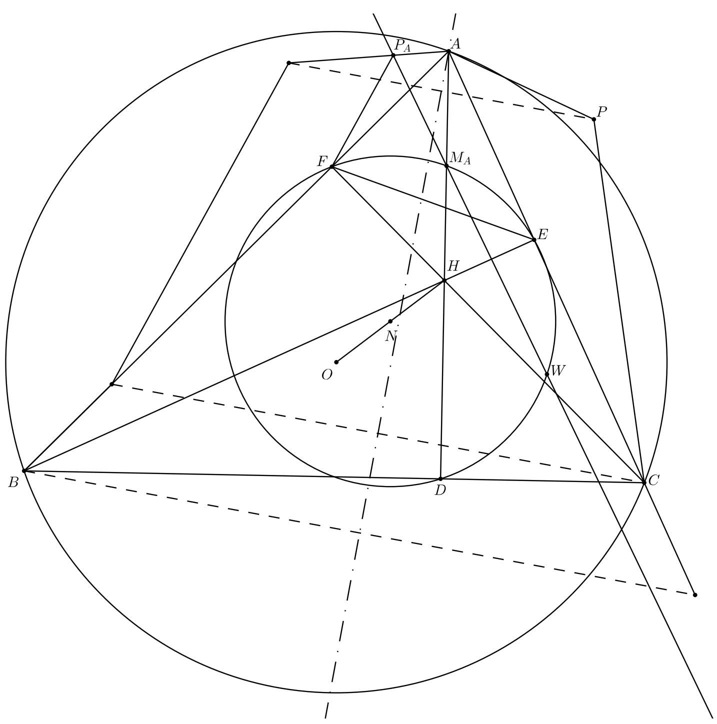
4471. *Proposed by Michael Diao.*

In $\triangle ABC$, let H be the orthocenter. Let M_A be the midpoint of AH and D be the foot from H onto BC, and define M_B, M_C, E and F similarly. Suppose P is a point in the plane distinct from the circumcenter of ΔABC, and suppose that P_A, P_B and P_C are points such that quadrilaterals $PABC$, P_AAEF, P_BDBF and P_CDEC are similar with vertices in that order. Show that M_AP_A, M_BP_B and M_CP_C concur on the circumcircle of $\triangle DEF$.

We received 5 submissions, all of which were correct. Only the proposer avoided the use of coordinates, so we provide two solutions: the proposer's together with one example of a solution via coordinates.

Solution 1, by Michel Bataille.

First, a few remarks (see figure):



(1) the circumcircle of ΔDEF is the nine-point circle whose centre is the midpoint N of OH (O being the circumcentre of ΔABC). Its radius is half the circumradius of ΔABC and it passes through M_A. We add the hypothesis that ΔABC is not right-angled (to keep the triangle DEF non-degenerate).

(2) The statement of the problem assumes as known that triangles ABC and AEF are oppositely similar, a fact for which a proof is as easily found as a reference: Points E, F, B, C lie on the circle with diameter BC (since $\angle BEC = \angle CFB = 90°$), hence, using directed angles modulo π, we have

$$\begin{aligned}\angle(EA, EF) &= \angle(EA, EB) + \angle(EB, EF) = \frac{\pi}{2} + \angle(CB, CF)\\ &= \frac{\pi}{2} + \angle(BC, BF) + \frac{\pi}{2}\\ &= \angle(BC, BA).\end{aligned}$$

Similarly, $\angle(FE, FA) = \angle(CA, CB)$, whence corresponding angles of triangles ABC and AEF are equal. The similarity σ such that $\sigma(A) = A, \sigma(B) = E, \sigma(C) = F$ is indirect (since $\angle(\sigma(CB), \sigma(CA)) = \angle(CA, CB)$); moreover, by definition $\sigma(P) = P_A$.

This said, we embed the problem in the complex plane and suppose without loss of generality that ΔABC is inscribed in the unit circle. We denote by a, b, c, m_a, p, p_a, n the affixes of A, B, C, M_A, P, P_A, N and we set $m = \frac{b+c}{2}$, the affix of the midpoint of BC. Note that $m \neq 0$ since ΔABC is not right-angled. We have

$$h = a + b + c, \quad n = \frac{h}{2} = \frac{a}{2} + m, \quad \text{and} \quad m_a = \frac{a+h}{2} = a + m.$$

The equation of the line AC is $z + ac\overline{z} = a + c$ and the equation of the perpendicular to AC through B is

$$\frac{z-b}{c-a} + \frac{\overline{z} - \overline{b}}{\overline{c} - \overline{a}} = 0,$$

that is, $bz - abc\overline{z} = b^2 - ac$ (using $\overline{a} = \frac{1}{a}$, etc.). From these equations, we deduce the affix e of E:

$$e = \frac{hb - ac}{2b}.$$

The similarity σ transforms the point with affix z into the point with affix $z' = \alpha\overline{z} + \beta$, where α, β satisfy

$$a = \alpha\overline{a} + \beta \quad \text{and} \quad \frac{hb - ac}{2b} = \alpha\overline{b} + \beta.$$

This yields $\alpha = -\frac{a(b+c)}{2} = -am$ and $\beta = a + m$ and therefore,

$$p_a = -am\overline{p} + a + m.$$

Now, let W (affix w) be the point of intersection distinct from M_A of the line M_AP_A and the nine-point circle: for some real number λ, we have

$$w = m_a + \lambda(m_a - p_a) = a + m + \lambda am\overline{p}.$$

Expressing that w must satisfy $|w-n|^2 = \frac{1}{4}$, we obtain

$$\lambda^2 m\overline{m}p\overline{p} + \frac{\lambda}{2}(m\overline{p} + \overline{m}p) = 0.$$

Since $w \neq m_a$, we have $\lambda \neq 0$, hence $\lambda = -\frac{1}{2}\left(\frac{1}{p\overline{m}} + \frac{1}{m\overline{p}}\right)$ and we readily deduce that

$$w = \frac{a+b+c}{2} - \frac{abc\overline{p}}{2p}.$$

Since the affix w is invariant under permutations of a, b, c, the lines M_BP_B and M_CP_C also pass through W and the required result follows.

Solution 2 by the proposer, revised by the editor.

We work in the Euclidean plane extended by the line at infinity. We shall be using properties of isogonal conjugation with respect to triangle $M_AM_BM_C$: Recall that the isogonal conjugate of a point with respect to $\Delta M_AM_BM_C$ is constructed by reflecting each line joining it to a vertex in the internal angle bisector at the corresponding vertex; the three reflected lines then concur at the isogonal conjugate. This type of conjugation is an involution of the points of the extended plane that are not on a sideline of the triangle; in particular, every point not on an extended side of the triangle is interchanged with its conjugate; moreover, each point other than a vertex on the circumcircle is interchanged with a point at infinity. Details can be found in standard textbooks such as Roger A. Johnson's *Advanced Euclidean Geometry* and Nathan Altshiller Court's *College Geometry*, as well as in standard internet sources.

Note that EF is antiparallel to BC with respect to $\angle BAC$ (where two lines are said to be *antiparallel with respect to an angle* if the image of either line under reflection in the angle bisector is parallel to the other line). For a proof, see the second preliminary remark in Solution 1 above. Because $PABC$ is similar to P_AAEF, the similarity that takes ΔABC to ΔAEF takes the circumcenter O of the former to the circumcenter M_A of the latter, whence the line OP and its image line M_AP_A must also be antiparallel with respect to $\angle BAC$. Since the triangles $M_AM_BM_C$ and ABC are homothetic, it follows that OP and M_AP_A are antiparallel with respect to $\angle M_BM_AM_C$. Letting P_∞ denote the point at infinity of the line OP, we have M_AP_∞ is parallel to OP, so we may conclude, finally, that M_AP_∞ and M_AP_A are isogonal in $\angle M_BM_AM_C$ (in the sense that $\angle M_BM_AM_C$ and $\angle P_AM_AP_\infty$ have the same angle bisectors).

Analogously, M_BP_∞ and M_BP_B are isogonal in $\angle M_CM_BM_A$, and M_CP_∞ and M_CP_C are isogonal in $\angle M_AM_CM_B$. Because these three lines (namely M_AP_∞, M_BP_∞ and M_CP_∞) meet on the line at infinity, their isogonal conjugates with respect to $\triangle M_AM_BM_C$, namely the lines M_AP_A, M_BP_B and M_CP_C, must concur at the isogonal conjugate of P_∞, which must lie on the circumcircle of $\triangle M_AM_BM_C$. But that circle is the nine-point circle of ΔABC, which coincides with the circumcircle of ΔDEF; in other words, the three lines concur at a point of the circumcircle of ΔDEF as claimed.

Editor's comments. The proposer observed that the concurrence point in question is the Poncelet point of the isogonal conjugate of P with respect to $\triangle ABC$. He also stated that by starting with P at the orthocenter, the result becomes,

> The Euler lines of $\triangle AEF$, $\triangle BFD$ and $\triangle CDE$ concur on the nine-point circle.

4472. *Proposed by Liam Keliher.*

Let n be a positive integer. Prove that n divides

$$\prod_{i=0}^{n-1}(2^n - 2^i).$$

We received 22 submissions, all correct. Several solvers proved a stronger result that the given product is actually divisible by $n!$. We present the solution of Prithwijit De.

Observe that $|\mathrm{GL}_n(\mathbb{Z}_2)| = \prod_{i=0}^{n-1}(2^n - 2^i)$ and the group of permutation matrices of order n (itself a group of order $n!$) is a subgroup of $\mathrm{GL}_n(\mathbb{Z}_2)$. Therefore

$$n!| \prod_{i=0}^{n-1}(2^n - 2^i).$$

Editor's Comment. Bataille pointed out that this result is well-known as it was problem 5 of the 4th National Mathematical Olympiad of Turkey, appearing in ***Crux*** before: see [2000 : 390] and [2002 : 503].

4473. *Proposed by Nguyen Viet Hung.*

Let $\lfloor a \rfloor$ denote the greatest integer not exceeding a. For every positive integer n,

(a) find the last digit of $\lfloor (2+\sqrt{3})^n \rfloor$,

(b) find $\gcd(\lfloor (2+\sqrt{3})^{n+1} \rfloor + 1, \lfloor (2+\sqrt{3})^n \rfloor + 1)$, where $\gcd(a, b)$ denotes the greatest common divisor of a and b.

We received 15 correct solutions. Most followed the tack of the general solution presented below.

(a) Let $u = 2+\sqrt{3}$ and $v_n = u^n + u^{-n}$ for $n \geq 1$. Then $0 < u^{-1} = 2-\sqrt{3} < 1$ and

$$v_n - 1 < u^n < v_n$$

for each positive integer n. Since u and u^{-1} are the roots of the quadratic equation $x^2 = 4x - 1$, the sequence $\{v_n\}$ satisfies the recursion

$$v_{n+2} = 4v_{n+1} - v_n$$

with initial conditions $v_1 = 4$ and $v_2 = 14$. Thus $\{v_n\}$ is a sequence of even integers and $\lfloor u^n \rfloor = v_n - 1$.

Modulo 10, $\{v_n\}$ has the period 3 cycle $\{4, 4, 2\}$, so that $\lfloor u^n \rfloor$ ends in the digit 3 when $n \equiv \pm 1 \pmod 3$ and in the digit 1 when $n \equiv 0 \pmod 3$.

(b) Since gcd $(v_1, v_2) = 2$ and $v_{n+2} = 4v_{n+1} - v_n$ for $n \geq 1$, an induction argument establishes that

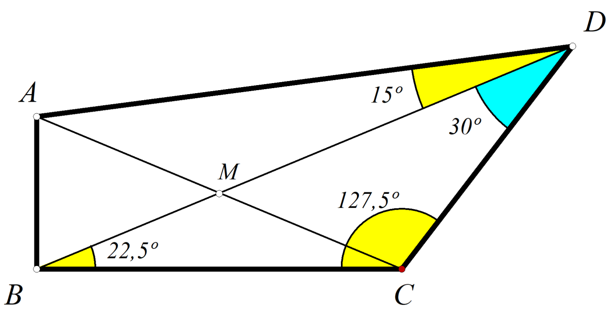
$$\gcd(\lfloor (2+\sqrt{3})^{n+1} \rfloor + 1, \lfloor (2+\sqrt{3})^n \rfloor + 1) = \gcd(v_{n+1}, v_n) = 2.$$

4474. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \frac{\pi}{2}$, $\angle ADB = \frac{\pi}{12}$, $\angle BDC = \frac{\pi}{6}$ and $\angle DBC = \frac{\pi}{8}$. Prove that BD passes through the midpoint of AC.

We received 23 submissions, all correct, and from the rich variety of solutions we have chosen two that were relatively light on the use of trigonometric identities.

Solution 1, by Cristobal Sánchez-Rubio.



Let M be the point of intersection of the lines AC and BD. Since ΔABC is a right triangle, it is enough to prove that the triangle MCB is isosceles; specifically, since we have $\angle DBC = \angle MBC = 22.5°$, we must show that $\angle MCB = 22.5°$. By the sine law applied to triangle ABD,

$$\frac{AB}{\sin 15°} = \frac{BD}{\sin 97.5°} = \frac{BD}{\cos 7.5°},$$

and therefore,

$$AB = \frac{BD \sin 15°}{\cos 7.5°}.$$

By the sine law for triangle BCD,

$$\frac{BC}{\sin 30°} = \frac{BD}{\sin 127.5°},$$

and therefore

$$BC = \frac{BD \sin 30°}{\cos 37.5°}.$$

Then

$$\frac{AB}{BC} = \frac{\sin 15^\circ \cdot \cos 37.5^\circ}{\sin 30^\circ \cos 7.5^\circ} = \frac{2\sin 7.5^\circ \cdot \cos 7.5^\circ \cdot \cos 37.5^\circ}{\sin 30^\circ \cos 7.5^\circ} = \frac{2\sin 7.5^\circ \cdot \cos 37.5^\circ}{\sin 30^\circ}.$$

But

$$\begin{aligned}2\sin 7.5^\circ \cdot \cos 37.5^\circ &= \sin(7.5^\circ + 37.5^\circ) + \sin(7.5^\circ - 37.5^\circ)\\ &= \sin 45^\circ - \sin 30^\circ\\ &= \frac{\sqrt{2}}{2} - \frac{1}{2},\end{aligned}$$

whence, finally,

$$\frac{AB}{BC} = \frac{2\sin 7.5^\circ \cdot \cos 37.5^\circ}{\sin 30^\circ} = \sqrt{2} - 1 = \tan 22.5^\circ.$$

In other words, $\angle MCB = 22.5^\circ$, so that $MB = MC$, as desired.

Solution 2, by Ivko Dimitrić.

Let M be the point of intersection of AC and BD, and let P and Q be the feet of the perpendiculars from A and C to BD, respectively. Then, from $\triangle ABP$ and $\triangle APD$ we have

$$\frac{BD}{AP} = \frac{BP}{AP} + \frac{PD}{AP} = \cot\frac{3\pi}{8} + \cot\frac{\pi}{12} = \tan\frac{\pi}{8} + \cot\frac{\pi}{12},$$

and from $\triangle BCQ$ and $\triangle CDQ$,

$$\frac{BD}{CQ} = \frac{BQ}{CQ} + \frac{QD}{CQ} = \cot\frac{\pi}{8} + \cot\frac{\pi}{6}.$$

Using the relevant half-angle formulas we have

$$\tan\frac{\pi}{8} = \sqrt{\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}} = \sqrt{\frac{(2-\sqrt{2})^2}{(2+\sqrt{2})(2-\sqrt{2})}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1$$

and $\cot\frac{\pi}{8} = \sqrt{2} + 1$, whereas

$$\cot\frac{\pi}{12} = \sqrt{\frac{1+\cos\frac{\pi}{6}}{1-\cos\frac{\pi}{6}}} = \sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}} = 2 + \sqrt{3}.$$

Then,

$$\tan\frac{\pi}{8} + \cot\frac{\pi}{12} = (\sqrt{2}-1) + (2+\sqrt{3}) = (\sqrt{2}+1) + \sqrt{3} = \cot\frac{\pi}{8} + \cot\frac{\pi}{6}.$$

Hence, $\dfrac{BD}{AP} = \dfrac{BD}{CQ}$, implying $AP = CQ$. As a consequence, the right triangles APM and CQM are congruent, implying $AM = CM$; that is, BD passes through the midpoint of AC.

4475. *Proposed by Michel Bataille.*

Let a, b be real numbers with $a, b, a+b, a-b \neq 0$. Prove the inequality

$$\frac{\sinh(2(a+b))}{a+b} + \frac{\sinh(2(a-b))}{a-b} \geq 4\left(\frac{\sinh^2(a)}{a} + \frac{\sinh^2(b)}{b}\right).$$

There were 4 correct and one incomplete solution. The correct solutions all used an integration argument along the lines of the following.

We first establish that

$$(\cosh x)(\cosh y) \geq \sinh x + \sinh y$$

for real x and y. This can be done either by using the identity $\cosh^2 t = 1 + \sinh^2 t$ and squaring, or by making the substitutions

$$(\sinh x, \cosh x) = (\tan u, \sec u) \quad \text{and} \quad (\sinh y, \cosh y) = (\tan v, \sec v)$$

with $\pi/2 < u, v < \pi/2$, and noting that the inequality is equivalent to

$$1 \geq \sin(u+v).$$

Let $(x, y) = (2ta, 2tb)$. For $0 \leq t \leq 1$,

$$\begin{aligned}\frac{1}{2}[\cosh(2t(a+b)) + \cosh(2t(a-b))] &= \cosh(2ta)\cosh(2tb) \\ &\geq \sinh(2ta) + \sinh(2tb).\end{aligned}$$

Integrate this inequality from 0 to 1 with respect to t to obtain

$$\begin{aligned}\frac{\sinh(2(a+b))}{4(a+b)} + \frac{\sinh(2(a-b))}{4(a-b)} &\geq \frac{\cosh(2a)-1}{2a} + \frac{\cosh(2b)-1}{2b} \\ &= \left(\frac{\sinh^2(a)}{a} + \frac{\sinh^2(b)}{b}\right),\end{aligned}$$

as desired.

4476. *Proposed by Leonard Giugiuc.*

Prove that for any real numbers a, b and c, we have

$$3\sqrt{6}(ab(a-b) + bc(b-c) + ca(c-a)) \leq ((a-b)^2 + (a-c)^2 + (b-c)^2)^{3/2}.$$

We received 10 submissions, all correct. We present a composite of nearly the same solutions by Michel Bataille and Marie-Nicole Gras.

Let

$$\begin{aligned}p &= \frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] = a^2 + b^2 + c^2 - ab - bc - ca, \\ q &= ab(a-b) + bc(b-c) + ca(c-a) = a^2b + b^2c + c^2a - ab^2 - bc^2 - ca^2.\end{aligned}$$

We need to prove that
$$3\sqrt{6}\,q \le (2p)^{\frac{3}{2}}. \quad (1)$$
If $q < 0$, then (1) clearly holds. If $q \ge 0$, then
$$(1) \iff 54q^2 \le (2p)^3 \iff 27q^2 \le 4p^3. \quad (2)$$
Consider the polynomial
$$\begin{aligned} P(x) &= (x-a+b)(x-b+c)(x-c+a) \\ &= x^3 + \big[(a-b)(b-c)+(b-c)(c-a)+(c-a)(a-b)\big]x - (a-b)(b-c)(c-a) \\ &= x^3 + (-a^2-b^2-c^2+ab+bc+ca)x - (-a^2b-b^2c-c^2a+ab^2+bc^2+ca^2) \\ &= x^3 - px + q, \end{aligned}$$
of discriminant
$$\Delta = -4p^3 + 27q^2.$$
Since $P(x)$ has 3 real roots by its definition, we must have $\Delta \le 0$, so $4p^3 - 27q^2 \ge 0$, from which (2) follows, completing the proof.

4477. *Proposed by Warut Suksompong.*

Given a positive integer n, let $a_1 \ge \cdots \ge a_n \ge 0$ and $b_1 \ge \cdots \ge b_n \ge 0$ be integers such that

1. $a_1 + \cdots + a_i \ge b_1 + \cdots + b_i$ for all $i = 1, \ldots, n-1$;
2. $a_1 + \cdots + a_n = b_1 + \cdots + b_n$.

Assume that there are n boxes, with box i containing a_i balls. In each move, Alice is allowed to take two boxes with an unequal number of balls, and move one ball from the box with more balls to the other box. Prove that Alice can perform a finite number of moves after which each box i contains b_i balls.

There were 4 correct solutions, all along the lines of the following.

The proof is by induction, the result being trivial for $n = 1$.

Assume it holds for at most $n-1$ boxes, with $n \ge 2$. If $a_j = b_j$ for some j with $1 \le j \le n$, we can remove box j from consideration. The conditions hold for the remaining $n-1$ boxes and we can invoke the induction hypothesis to rearrange the balls among them.

Henceforth, let $a_i \ne b_i$ for each i.

Since $a_1 > b_1$ and
$$a_n = b_n - [(a_1 + \cdots + a_{n-1}) - (b_1 + \cdots + b_{n-1})] \le b_n,$$
then $a_n < b_n$ and there is a positive integer $k \le n-1$ for which $a_k > b_k$ and $a_{k+1} < b_{k+1}$. Start removing balls one at a time from box k and placing them in box $k+1$.

If $a_k - b_k \le b_{k+1} - a_{k+1}$, then transfer a total of $a_k - b_k$ balls from box $k+1$, leaving b_k balls in box k and

$$a_{k+1} + a_k - b_k \le a_{k+1} + b_{k+1} - a_{k+1} = b_{k+1}$$

balls in box $k+1$.

Since

$$(a_1 + \cdots + a_{k-1}) + b_k + (a_{k+1} + a_k - b_k) = a_1 + \cdots + a_{k+1} \ge b_1 + \cdots + b_{k+1},$$

we see that the consequent arrangement of balls in the boxes satisfies the two conditions.

If $a_k - b_k > b_{k+1} - a_{k+1}$, transfer $b_{k+1} - a_{k+1}$ balls from box k to box $k+1$, leaving

$$a_k - (b_{k+1} - a_{k+1}) > a_k - (a_k - b_k) = b_k$$

balls in box k and b_{k+1} balls in box $k+1$.

Since

$$(a_1 + \cdots + a_{k-1}) + [a_k - (b_{k+1} - a_{k+1})] > (b_1 + \cdots + b_{k-1}) + b_k$$

and

$$\begin{aligned}(a_1 + \cdots + a_{k-1}) + [a_k - (b_{k+1} - a_{k+1})] + b_{k+1} &= a_1 + \cdots + a_{k-1} + a_k + a_{k+1} \\ &\ge b_1 + \cdots + b_{k+1},\end{aligned}$$

the consequent arrangement of balls in the boxes satisfies the conditions.

In either case, we have a rearrangement of balls for which the number of balls in one of the kth and $(k+1)$th boxes is equal to the corresponding value of b, so we can apply the induction step.

4478. *Proposed by Florin Stanescu.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a^2) - f(b^2) \le (f(a) + b)(a - f(b))$$

for all $a, b \in \mathbb{R}$.

There were 11 correct and 3 incorrect submitted solutions.

The only solutions are $f(x) = x$ and $f(x) = -x$.

Make the substitutions $(a, b) = (x, y)$ and $(a, b) = (y, x)$ to obtain

$$f(x^2) - f(y^2) \le xy - f(x)f(y) + xf(x) - yf(y)$$

and

$$f(y^2) - f(x^2) \le xy - f(x)f(y) - xf(x) + yf(y).$$

Adding these inequalities leads to $f(x)f(y) \le xy$ for all real x and y. In particular, $f(0)^2 \le 0$, so that $f(0) = 0$.

Substituting $(a,b) = (x,0)$ and $(a,b) = (0,x)$ yields

$$f(x^2) \le xf(x) \quad \text{and} \quad -f(x)^2 \le -xf(x),$$

whence $f(x^2) = xf(x)$ for all real x. In particular, $f(1) = -f(-1)$.

Since

$$f(1)f(x) \le x \quad \text{and} \quad -f(1)f(x) = f(-1)f(x) \le -x,$$

then $f(1)f(x) = x$ and $f(1)^2 = 1$. When $f(1) = 1$, then $f(x) \equiv x$, and when $f(1) = -1$, then $f(x) \equiv -x$.

4479. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with $\angle A = 90^\circ$ and let H be the foot of the altitude from A. Prove that

$$\frac{6}{(AB+AC)^2} - \frac{1}{2 \cdot AH^2} \le \frac{1}{BC^2}.$$

We received 32 solutions, including two from the featured solver. We present the solution by Miguel Amengual Covas.

Denote the length of the hypotenuse of the given triangle by a and the legs by b and c. Then the area of ΔABC may be expressed as $bc/2$, and also as $a \cdot AH/2$. Equating these and solving for AH, we get

$$AH = \frac{bc}{a}.$$

When this is substituted into the proposed inequality, the proposed inequality becomes

$$\frac{6}{(b+c)^2} - \frac{a^2}{2b^2c^2} \le \frac{1}{a^2}. \tag{1}$$

We subsitute $b^2 + c^2$ for a^2 in (1), obtaining

$$\frac{6}{(b+c)^2} - \frac{b^2+c^2}{2b^2c^2} \le \frac{1}{a^2},$$

or, equivalently,

$$b^6 + 2b^5c - 7b^4c^2 + 8b^3c^3 - 7b^2c^4 + 2bc^5 + c^6 \ge 0.$$

This in turn is equivalent to

$$(b-c)^2\left(b^4 + 4b^3c + 4bc^3 + c^4\right) \ge 0,$$

whose validity is obvious. Equality occurs if and only if $b = c$.

4480. *Proposed by Leonard Giugiuc.*

Find all the solutions to the system

$$\begin{cases} a+b+c+d=4, \\ a^2+b^2+c^2+d^2=6, \\ a^3+b^3+c^3+d^3=\frac{94}{9}, \end{cases}$$

in $[0,2]^4$.

We received 11 submissions, all correct. We present the solution by Digby Smith.

Note first that

$$\begin{aligned} ab+ac+ad+bc+bd+cd &= \frac{1}{2}\left((a+b+c+d)^2-(a^2+b^2+c^2+d^2)\right) \\ &= \frac{1}{2}(4^2-6)=5, \end{aligned} \tag{1}$$

and

$$\begin{aligned} (a+b+c+d)^3 &= a^3+b^3+c^3+d^3+3a^2(b+c+d)+3b^2(c+d+a) \\ &\quad +3c^2(d+a+b)+3d^2(a+b+c)+6(abc+abd+acd+bcd) \\ &= 3(a+b+c+d)(a^2+b^2+c^2+d^2)-2(a^3+b^3+c^3+d^3) \\ &\quad +6(abc+abd+acd+bcd) \\ \implies 64 &= 3(4)(6)-2\frac{94}{9}+6(abc+abd+acd+bcd) \\ \implies abc+abd+acd+bcd &= \frac{1}{6}(64+\frac{188}{9}-72)=\frac{58}{27}. \end{aligned} \tag{2}$$

Next, let $k=abcd$ and $p(x)$ be the polynomial function defined by

$$p(x)=(x-a)(x-b)(x-c)(x-d).$$

Then by (1) and (2) we have

$$\begin{aligned} p(x) &= x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2 \\ &\quad -(abc+bcd+cda+dab)x+abcd \\ &= x^4-4x^3+5x^2-\frac{58}{27}x+k. \end{aligned}$$

Since

$$\begin{aligned} p'(x) &= 4x^3-12x^2+10x-\frac{58}{27} \\ &= \frac{1}{27}(108x^3-324x^2+270x-58) \\ &= \frac{2}{27}(3x-1)(18x^2-48x+29), \end{aligned}$$

solving $p'(x) = 0$ yields

$$x = \frac{1}{3},\ \frac{4}{3} \pm \frac{\sqrt{6}}{2}.$$

Since $p(x)$ is a 4th degree polynomial with positive leading coefficient and $p'(x)$ has 3 distinct real roots in (0,2), it follows that in order for a, b, c, d to be solutions of the given equations, where $0 \le a, b, c, d \le 2$, we must have

$$p(0) \ge 0,\ p(\frac{1}{3}) \le 0,\ p(\frac{4}{3} - \frac{\sqrt{6}}{2}) \ge 0,\ p(\frac{4}{3} + \frac{\sqrt{6}}{2}) \le 0,\ p(2) \ge 0.$$

Evaluating, we find $p(\frac{1}{3}) = p(2) = k - \frac{8}{27}$. Hence, $k = \frac{8}{27}$, from which we obtain

$$\begin{aligned} p(x) &= x^4 - 4x^3 + 5x^2 - \frac{58}{27}x + \frac{8}{27} \\ &= \frac{1}{27}(27x^4 - 108x^3 + 135x^2 - 58x + 8) \\ &= \frac{1}{27}(3x-1)^2(3x-4)(x-2). \end{aligned}$$

Therefore, the solutions in $[0,2]^4$ are the 12 permutations of $(\frac{1}{3}, \frac{1}{3}, \frac{4}{3}, 2)$.