OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(8), p. 463–465.

OC446. Given the numbers $2, 3, \ldots, 2017$ and the natural number $n \leq 2014$, Ivan and Peter play the following game: Ivan selects n numbers from the given ones, then Peter selects 2 numbers from the remaining numbers, then all the selected $n+2$ numbers are ranked in value:

$$a_1 < a_2 < \ldots < a_{n+2}.$$

If there exists i, $1 \leq i \leq n+1$ for which a_i divides a_{i+1}, then Peter wins, otherwise Ivan wins. Find all n for which Ivan has a winning strategy.

Originally Bulgaria Math Olympiad, 3rd Problem, Grade 11, Second Round 2017.

We received 1 submission. We present the solution by Oliver Geupel.

We show that Ivan has a winning strategy if and only if $n \geq 10$.

First, suppose $n \geq 10$. Consider the intervals $I_k = \left[2^{k-1}+1,\ 2^k\right]$, where $1 \leq k \leq 10$, and $I_{11} = \left[2^{10}+1,\ 2017\right]$. Note that $\{2, 3, \ldots, 2017\} = \bigcup_{k=1}^{11} I_k$. Every interval has the property that its upper endpoint is less than twice its lower endpoint. Hence, for any two distinct integers in the same interval, it cannot happen that one of them divides the other one. From this observation, it follows that Ivan wins if his selection includes the lower endpoints of all intervals (except for the number 2). As consequence, it is enough for Ivan to include the ten numbers 3, 5, 9, 17, 33, 65, 129, 257, 513, and 1025 in his selection.

It remains to show that Ivan has no winning strategy when $n < 10$. Consider the disjoint intervals $J_k = \left[2^{k-1}+2,\ 2^k+1\right]$ where $1 \leq k \leq 10$. By the pigeonhole principle, there exists an index k such that J_k is disjoint to Ivan's selection. So Peter can avoid choosing any numbers from the interval J_k which Ivan has already avoided, and can choose the numbers $2^{k-1}+1$ and 2^k+2 (if not selected by Ivan). Thus these two numbers become a_i and a_{i+1} for some i, so Peter wins the game.

OC447. Let $m > 1$ be an integer and let $N = m^{2017}+1$. Positive numbers $N, N-m, N-2m, \ldots, m+1, 1$ are written in a row. At each step, the leftmost number and all of its divisors (if any) are erased. This process continues until all the numbers are erased. What are the numbers deleted at the last step?

Originally Bulgaria Math Olympiad, 2nd Problem, Grades 9-12, Final Round 2017.

We received 1 submission. We present the solution by Oliver Geupel.

We prove that the number deleted at the last step is

$$M = \frac{N}{m+1} + m = m^{2016} + 1 - m^2(m-1)\sum_{k=0}^{1006} m^{2k}.$$

First, $M \equiv 1 \pmod{m}$ and $M \leq N$, which implies that M is a member of the row. Second, M is not a proper divisor of any number in the row, because numbers $2M$, $3M$, $4M$, $\ldots$, mM are not congruent to $1 \pmod{m}$, and $(m+1)M > N$. Thus, the number M is the leftmost remaining number at some step.

Next, we show that all numbers in the row that are strictly smaller than M are erased before M. Let $r < M$ be a number in the row. It is enough to prove that there exists a positive integer k such that the number $(km+1)r$ is in the row and $(km+1)r > M$; consequently, r is erased before M. Equivalently, we have to show that at least one member of the arithmetic progression $\{r + kmr\}_{k=1,2,\ldots}$ is in the interval $(M, N]$. For example, for the largest $r < M$ in the row, $r = N/(m+1)$, it is enough to select $k = 1$: $(km+1)r = N$. For an arbitrary $r < N/(m+1)$ in the row

$$mr \leq \frac{m}{m+1} \cdot N - m = N - \left(\frac{N}{m+1} + m\right) = N - M.$$

Therefore, the increment of arithmetic progression, $\{r + kmr\}_{k=1,2,\ldots}$, is strictly smaller than the length of interval $(M, N]$, and at least one member of the arithmetic progression must belong to interval $(M, N]$.

Hence M is the number erased at the last step.

OC448. Let $x_1 \leq x_2 \leq \ldots \leq x_{2n-1}$ be real numbers whose arithmetic mean is equal to A. Prove that

$$2\sum_{i=1}^{2n-1} (x_i - A)^2 \geq \sum_{i=1}^{2n-1} (x_i - x_n)^2 .$$

Originally Poland Math Olympiad, 3rd Problem, Second Round 2017.

No solutions were received.

OC449. A sequence $(a_1, a_2, \ldots, a_k)$ consisting of pairwise distinct squares of an $n \times n$ chessboard is called a *cycle* if $k \geq 4$ and the squares a_i and a_{i+1} have a common side for all $i = 1, 2, \ldots, k$, where $a_{k+1} = a_1$. Subset X of this chessboard's squares is *mischievous* if each cycle on it contains at least one square in X. Determine all real numbers C with the following property: for each integer $n \geq 2$, on an $n \times n$ chessboard there exists a mischievous subset consisting of at most Cn^2 squares.

Originally Poland Math Olympiad, 2nd Problem, Final Round 2017.

No solutions were received.

OC450. Find all pairs (x, y) of real numbers satisfying the system of equations

$$x \cdot \sqrt{1-y^2} = \frac{1}{4}\left(\sqrt{3}+1\right),$$
$$y \cdot \sqrt{1-x^2} = \frac{1}{4}\left(\sqrt{3}-1\right).$$

Originally Germany Math Olympiad, 3rd Problem, Grades 11-12, Second Day, 3rd Round 2017.

We received 19 submissions. We present two solutions.

Solution 1, by the Missouri State University Problem Solving Group.

Suppose x, y are solutions. From the given equations, $0 < x, y < 1$. So we may set $x = \sin\alpha$ and $y = \sin\beta$ for some $0 < \alpha, \beta < \pi/2$. Then

$$4\sin\alpha\cos\beta = \sqrt{3}+1,$$

$$4\cos\alpha\sin\beta = \sqrt{3}-1.$$

Add and subtract the two equations and divide by 4 to get

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sqrt{3}/2,$$

$$\sin\alpha\cos\beta - \cos\alpha\sin\beta = 1/2.$$

Hence solving the original system reduces to solving

$$\sin(\alpha+\beta) = \sqrt{3}/2,$$

$$\sin(\alpha-\beta) = 1/2,$$

with $0 < \alpha, \beta < \pi/2$. The angles α and β are given by

$$\alpha+\beta = \pi/3 + 2\pi n, \qquad \alpha-\beta = \pi/6 + 2\pi m$$

$$\alpha+\beta = 2\pi/3 + 2\pi n, \qquad \alpha-\beta = \pi/6 + 2\pi m$$
$$\alpha+\beta = \pi/3 + 2\pi n, \qquad \alpha-\beta = 5\pi/6 + 2\pi m$$
$$\alpha+\beta = 2\pi/3 + 2\pi n, \qquad \alpha-\beta = 5\pi/6 + 2\pi m$$

or by

$$\alpha = \pi/4 + (n+m)\pi, \qquad \beta = \pi/12 + (n-m)\pi,$$
$$\alpha = 5\pi/12 + (n+m)\pi, \qquad \beta = \pi/4 + (n-m)\pi,$$
$$\alpha = 7\pi/12 + (n+m)\pi, \qquad \beta = -\pi/4 + (n-m)\pi,$$
$$\alpha = 3\pi/4 + (n+m)\pi, \qquad \beta = -\pi/12 + (n-m)\pi.$$

for some integers n, m. Since $0 < \alpha, \beta < \pi/2$, then either $\alpha = \pi/4$, $\beta = \pi/12$, or $\alpha = 5\pi/12$, $\beta = \pi/4$. The two solutions of the initial system are

$$(x, y) = \left(\sin\frac{\pi}{4},\ \sin\frac{\pi}{12}\right) = \left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{6}-\sqrt{2}}{4}\right)$$

and

$$(x,y)=\left(\sin\frac{5\pi}{12},\ \sin\frac{\pi}{4}\right)=\left(\frac{\sqrt{6}+\sqrt{2}}{4},\ \frac{\sqrt{2}}{2}\right).$$

Solution 2, by David Manes.

The two pairs (x,y) of real numbers that satisfy the system of equations are

$$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2-\sqrt{3}}}{2}\right)\quad\text{and}\quad\left(\frac{\sqrt{2+\sqrt{3}}}{2},\frac{\sqrt{2}}{2}\right).$$

We can check that these values of x and y satisfy the two equations.

Since $1-y^2\geq 0$ and $1-x^2\geq 0$ it follows that $-1\leq x,y\leq 1$. Moreover, $(\sqrt{3}+1)/4>0$, $(\sqrt{3}-1)/4>0$, $\sqrt{1-x^2}>0$, and $\sqrt{1-y^2}>0$. Therefore, $x,y>0$. Hence, if x and y solve the system then $0<x,y<1$. Squaring each of the two equations, we obtain

$$x^2\left(1-y^2\right)=\frac{1}{8}\left(2+\sqrt{3}\right),$$
$$y^2\left(1-x^2\right)=\frac{1}{8}\left(2-\sqrt{3}\right).$$

Adding and then subtracting the two equations yields

$$2x^2+2y^2-4x^2y^2=1,$$
$$x^2-y^2=\frac{\sqrt{3}}{4}.$$

From the second equation, we obtain $y^2=x^2-\sqrt{3}/4$. Rearranging the terms in the first equation, we obtain $2x^2\left(1-2y^2\right)=1-2y^2$. Hence, either $x^2=1/2$ or $1-2y^2=0$.

First, if $x=\pm\sqrt{2}/2$ then $y^2=x^2-\sqrt{3}/4=(2-\sqrt{3})/4$, and $y=\pm\sqrt{2-\sqrt{3}}/2$. Note that $\sqrt{2}/2$ and $\sqrt{2-\sqrt{3}}/2$ belong to the interval $(0,1)$. We obtain the first solution $x=\sqrt{2}/2$ and $y=\sqrt{2-\sqrt{3}}/2$.

Second, if $1-2y^2=0$, then $y=\pm\sqrt{2}/2$ and $x^2=y^2+\sqrt{2}/4=(2+\sqrt{3})/4$. Therefore, $x=\pm\sqrt{2+\sqrt{3}}/2$. Since $0<x,y<1$, we find the second solution $x=\sqrt{2+\sqrt{3}}/2$ and $\sqrt{2}/2$. This solves the system.

Editor's comments. All submissions followed one of the two techniques presented above: trigonometric or algebraic approach.