MathemAttic

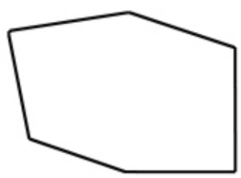
No. 13

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **May 15, 2020**.

MA61. A hexagon has consecutive angle measures of 90°, 120°, 150°, 90°, 120° and 150°. If all of its sides are 4 units in length, what is the area of the hexagon?



MA62. A positive integer n is called "savage" if the integers $\{1, 2, \ldots, n\}$ can be partitioned into three sets A, B and C such that

i) the sum of the elements in each of A, B and C is the same,

ii) A contains only odd numbers,

iii) B contains only even numbers, and

iv) C contains every multiple of 3 (and possibly other numbers).

Now consider the following:

(a) Show that 8 is a savage integer.

(b) Prove that if n is an even savage integer, then $\frac{n+4}{12}$ is an integer.

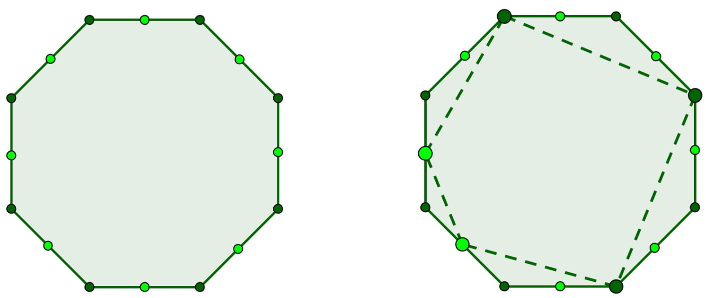
MA63. One way to pack a 100 by 100 square with 10 000 circles, each of diameter 1, is to put them in 100 rows with 100 circles in each row. If the circles are repacked so that the centres of any three tangent circles form an equilateral triangle, what is the maximum number of additional circles that can be packed?

MA64. A regular octagon is shown in the first diagram below, with the vertices and midpoints of the sides marked.

An "inner polygon" is a polygon formed by traversing the octagon in a clockwise manner, selecting some of the marked points as you go, ensuring that each side of the original octagon contains exactly one selected point. Then each selected point is connected to the next with a line segment, and the last is connected to the first to complete the inner polygon.

An example of an inner polygon is shown in the second diagram.

How many inner polygons does the regular octagon have?



MA65. There are four unequal, positive integers a, b, c, and N such that $N = 5a + 3b + 5c$. It is also true that $N = 4a + 5b + 4c$ and N is between 131 and 150. What is the value of $a + b + c$?

. . . .

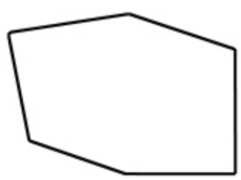
Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mai 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA61. Un hexagone a des angles, dans l'ordre, de 90°, 120°, 150°, 90°, 120° et 150°. Si tous les côtés sont de longueur 4, quelle est la surface de l'hexagone?



MA62. Un entier positif n est dit "sauvage" si les entiers $\{1, 2, \ldots, n\}$ peuvent être partitionnés en trois ensembles A, B et C de façon à ce que

i) les sommes des éléments dans A, B et C sont les mêmes,

ii) A contient seulement des entiers impairs,

iii) B contient seulement des entiers pairs et

iv) C contient tous les multiples de 3 (et possiblement d'autres nombres).

Alors:

(a) Démontrer que 8 est un entier sauvage.

(b) Démontrer que si n est un entier sauvage pair, alors $\dfrac{n+4}{12}$ est un entier.

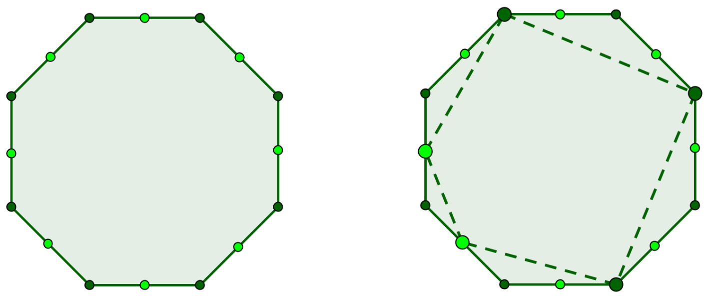
MA63. Une façon de placer 10,000 cercles de diamètre 1 dans un carré de taille 100 par 100 serait de placer 100 cercles dans chacune des 100 rangées. Si par contre on replace les cercles de façon à ce que les centres de trois cercles tangents forment un triangle équilatéral, quel est le nombre maximum de cercles additionnels pouvant être placés?

MA64. Un octagone régulier est indiqué au premier diagramme ci-bas, où sont marqués les sommets et les mi points des côtés.

Un "polygone interne" est un polygone formé en parcourant l'octagone dans le sens des aiguilles d'une montre, choisissant certains des points marqués, tout en s'assurant que chaque côté de l'octagone original contient exactement un point choisi. Et puis, chaque point choisi est relié au prochain avec un segment de ligne, le dernier étant relié au premier.

Un exemple d'un polygone interne est indiqué au deuxième diagramme.

Combien de polygones internes l'octagone a-t-il ?



MA65. Soient a, b, c et N, quatre entiers positifs distincts tels que $N = 5a + 3b + 5c$. De plus, $N = 4a + 5b + 4c$ et N se situe entre 131 et 150. Quelle est la valeur de $a + b + c$?

MathemAttic
SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(8), p. 447–449.

MA36. Let A and B be sets with the property that there are exactly 144 sets which are subsets of at least one of A or B. How many elements does the union of A and B have?

Originally problem 13 of "A Mathematical Orchard: Problems and Solutions" by Mark I. Krusemeyer, George T. Gilbert, and Loren C. Larson.

We received 5 submissions, of which 3 were correct and complete. We present the solution by Digby Smith, slightly modified by the editor.

Let M be the number of elements in A and N the number of elements in B; without loss of generality, assume $N \leq M$. Let K be the number of elements in the intersection of A and B, and note $0 \leq K \leq N$.

The number of subsets (including the empty set) of a set with n distinct elements is 2^n, so the number of sets which are subsets of at least one of A and B is $2^M + 2^N - 2^K$. We have

$$144 = 2^M + 2^N - 2^K < 2^M + 2^M = 2^{M+1}$$

which gives us $8 \leq M + 1$, so $7 \leq M$. On the other hand,

$$144 = 2^M + 2^N - 2^K \geq 2^M,$$

so $7 \geq M$, and we conclude that $M = 7$.

Hence there are $144 - 128 = 16$ nonempty subsets which are subsets of B but not of A. There are $2^N - 2^K$ sets which are subsets of B but not of A, so we must have $N > K$ and also

$$2^K(2^{(N-K)} - 1) = 2^4.$$

If $N - K > 1$ then $2^{(N-K)} - 1$ is an odd number greater than 1, which cannot divide 2^4. Thus we must have $K = 4$ and $N = 5$. Hence the number of elements in the union of A and B is $M + N - K = 8$.

MA37. Both 4 and 52 can be expressed as the sum of two squares as well as exceeding another square by 3:

$$4 = 0^2 + 2^2 \quad \text{and} \quad 4 - 3 = 1^2,$$
$$52 = 4^2 + 6^2 \quad \text{and} \quad 52 - 3 = 7^2.$$

Show that there are an infinite number of such numbers that have these two characteristics.

Originally problem 123 of "A Mathematical Orchard: Problems and Solutions" by Mark I. Krusemeyer, George T. Gilbert, and Loren C. Larson.

We received 7 submissions, of which 6 were correct and complete. We present the solution by Anita Hessami Pilehrood.

We need to show that there is a k such that $k = a^2 + b^2$ and $k = c^2 + 3$ for nonnegative integers a, b, c. Let $a = 2n$, $b = 2n^2 - 2$, and $c = 2n^2 - 1$ for a positive integer n. Then

$$a^2 + b^2 = 4n^2 + 4n^4 + 4 - 8n^2 = 4n^4 - 4n^2 + 4$$

and

$$c^2 + 3 = 4n^4 + 1 - 4n^2 + 3 = 4n^4 - 4n^2 + 4.$$

Therefore $a^2 + b^2 = c^2 + 3$ and hence there is an infinite sequence of numbers $k = 4(n^4 - n^2 + 1)$, $n = 1, 2, 3, \ldots$, that have these two characteristics.

MA38. Consider a 12×12 chessboard consisting of 144 1×1 squares. If three of the four corner squares are removed, can the remaining area be covered by placing 47 1×3 tiles?

Originally problem 33 of "A Mathematical Orchard: Problems and Solutions" by Mark I. Krusemeyer, George T. Gilbert, and Loren C. Larson.

We received 4 solutions, of which 3 were complete and correct. We present the solution by the Missouri State University Problem Solving Group.

Number the squares of the (intact) chessboard as shown in the figure.

3	1	2	3	1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2	3	1	2	3
3	1	2	3	1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2	3	1	2	3
3	1	2	3	1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2	3	1	2	3
3	1	2	3	1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2	3	1	2	3

Each number appears 48 times and any 1×3 tile will cover exactly one of each number, regardless of its position or orientation. Therefore, in order to tile the board with three corner squares removed, we must remove one of each square with a given number. In particular, we cannot remove the upper-left and lower-right corner squares, which are both labeled 3. Similarly, rotating the numbering scheme 90°, we conclude that we cannot remove the lower-left and upper-right corner squares. But removing any three corner squares will result in two diagonally opposite corners being removed and this gives us a contradiction.

MA39. Point E is selected on side AB of triangle ABC in such a way that $AE : EB = 1 : 3$ and point D is selected on side BC so that $CD : DB = 1 : 2$. The point of intersection of AD and CE is F. Determine the value of $\frac{EF}{FC} + \frac{AF}{FD}$.

Originally MAA Problem Book II (1961–1965), Question 37, 1965 examination.

We received 8 submissions, all correct. We present the solution provided by Anita Hessami Pilehrood.

Let the area of ΔCFD be y. Then $[DFB] = 2y$ since $\dfrac{DB}{DC} = 2$ and ΔDFB and ΔCFD share the height dropped from vertex F. Similarly, let the area of ΔAFE be x. Then the area of ΔEFB equals $3x$ since $\dfrac{EB}{AE} = 3$ and both triangles have a common height dropped from vertex F.

Now let's consider ΔBCE and ΔECA. Since $\dfrac{BE}{AE} = 3$ and these two triangles have a common height from vertex C, we have $\dfrac{[BCE]}{[ECA]} = 3$ which implies

$$\frac{y + 2y + 3x}{[CFA] + x} = 3,$$

and thus $[CFA] = y$.

Similarly in ΔBAD and ΔCAD, we have $\dfrac{BD}{DC} = 2$ and a common height from vertex A. Thus,

$$\frac{[BAD]}{[CAD]} = 2 \quad \text{implies} \quad \frac{4x + 2y}{2y} = 2, \quad \text{so} \quad \frac{x}{y} = \frac{1}{2}.$$

We have

$$\frac{EF}{FC} = \frac{[EAF]}{[FAC]} = \frac{x}{y} = \frac{1}{2},$$
$$\frac{AF}{FD} = \frac{[ACF]}{[FCD]} = \frac{y}{y} = 1,$$

and therefore

$$\frac{EF}{FC} + \frac{AF}{FD} = \frac{1}{2} + 1 = \frac{3}{2}.$$

MA40. In racing over a given distance d at uniform speeds, A can beat B by 20 yards, B can beat C by 10 yards, and A can beat C by 28 yards. Determine the distance d in yards.

Originally MAA Problem Book II (1961–1965), Question 37, 1961 examination.

We received 7 submissions, all of which were complete and correct. We present the solution of Aaratrick Basu, lightly edited.

Let v_A be the speed of A, v_B be the speed of B, and v_C be the speed of C.

As per the problem, A beats B by 20 yards, i.e,

$$\frac{d}{v_A} = \frac{d-20}{v_B} \tag{1}$$

Similarly, we have

$$\frac{d}{v_B} = \frac{d-10}{v_C}, \tag{2}$$
$$\frac{d}{v_A} = \frac{d-28}{v_C}. \tag{3}$$

Then from (1) and (3) we get

$$\frac{d-20}{v_B} = \frac{d-28}{v_C} \implies \frac{v_B}{v_C} = \frac{d-20}{d-28}.$$

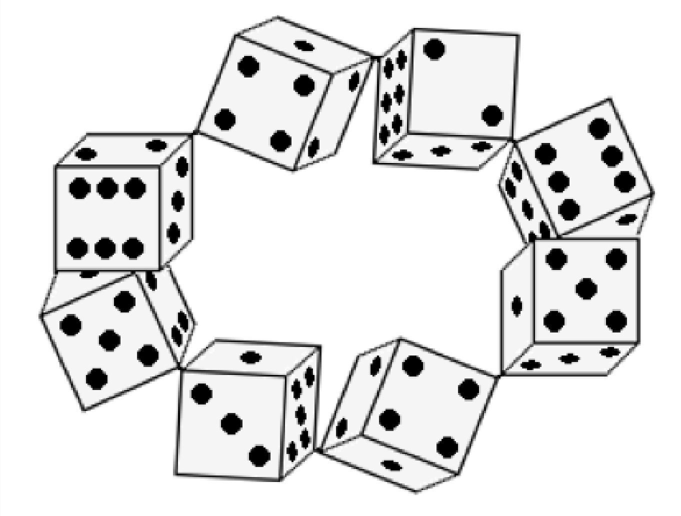
With (2) this becomes

$$\frac{d}{d-10} = \frac{d-20}{d-28} \implies d^2 - 30d + 200 = d^2 - 28d,$$

which reveals $d = 100$. Hence, we have that A, B, C were racing over $d = 100$ yards.

Bracelet made of cubes

Take eight unit cubes, or playing dice, and mark them with dots so that the sums of dots on opposite faces are all equal to 7 (that is, the opposite faces have 1 and 6 dots, 2 and 5 dots, 3 and 4 dots). Then, for each cube, drill an all-the-way-through diagonal hole from the vertex where faces with 1, 2 and 3 dots meet to the vertex where faces with 4, 5 and 6 dots meet. Take a strong thread and string all 8 cubes together through their holes in the direction they were drilled. Tie the thread to get a beautiful bracelet made of cubes:



Now, perform the following tasks:

1. fold this bracelet into a $2 \times 2 \times 2$ cube;
2. fold this bracelet into a $2 \times 2 \times 2$ cube so that the sum of dots on each of its faces is 14;
3. prove that you cannot fold this bracelet into a $2 \times 2 \times 2$ cube so that the sum of dots on each of its faces is 13.

Puzzle by Nikolai Avilov.

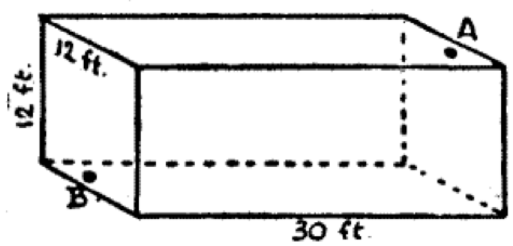
TEACHING PROBLEMS

No.9

Erick Lee

The Spider and the Fly

Inside a rectangular room, measuring 30 feet in length and 12 feet in width and height, a spider is at a point on the middle of one of the end walls, 1 foot from the ceiling, as at A; and a fly is on the opposite wall, 1 foot from the floor in the centre, as shown at B. What is the shortest distance that the spider must crawl in order to reach the fly, which remains stationary? Of course the spider never drops or uses its web, but crawls fairly.

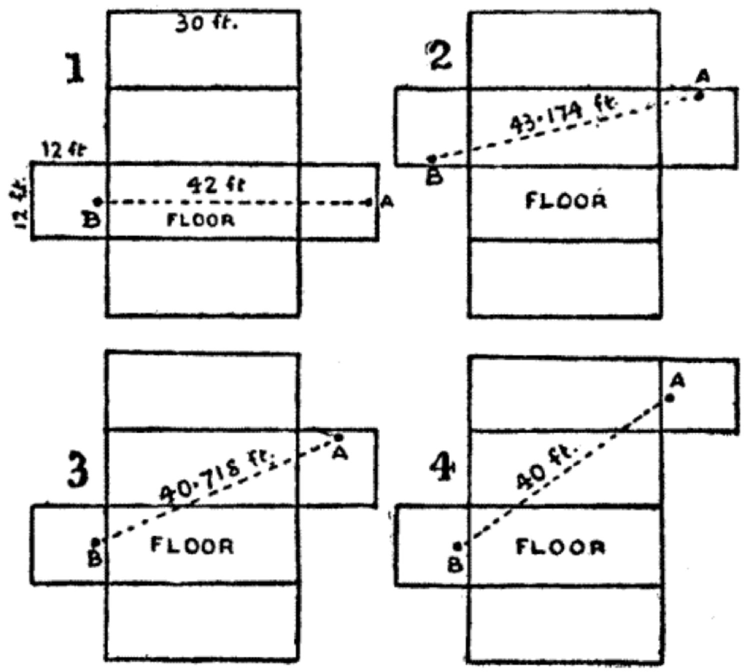


This problem was created by Henry Ernest Dudeney. It is problem 75 from his book *The Canterbury Puzzles* published in 1908. Dudeney was an English mathematician and prolific creator of logic puzzles and recreational mathematics problems. From 1910 until his death in 1930, Dudeney wrote a monthly column in *The Strand Magazine* entitled "Perplexities" which featured mathematical brain-teasers. As *The Canterbury Puzzles* was published over 100 years ago, it is freely available on The Project Gutenberg website at `http://www.gutenberg.org/files/27635/27635-h/27635-h.htm`.

When introducing this problem to students, I draw a spider and a fly each on their own index card and tape them to the appropriate spots on the wall in the classroom which is the shape of a rectangular prism, although not exactly the same dimensions as the given problem. I then describe the problem of the spider and the fly using the classroom to physically model the problem. Students often struggle to visualize problems in three dimensions despite living in a three-dimensional world. Many students will quickly determine that the spider should take the "straight path" directly up to the ceiling (1 ft), directly across the ceiling (30 ft), and down the opposite wall (11 ft). This will give a total distance of 42 ft.

After the majority of the class has come to this conclusion, we have a discussion. I ask them, "How do you know that this is the shortest path?" Students often respond that a straight line is the shortest distance between two points. To challenge their thinking, we discuss how a straight line might look different in three dimensions than in two dimensions. The great circle routes that airplanes fly often seem counterintuitive when students visualize the Mercator projection maps commonly found in classrooms. Where we live in Nova Scotia, we can look into the sky at almost any time of the day and see airplanes high in the sky flying from the Northeastern United States to European destinations. This only makes sense when looking at the great circle route on a spherical globe.

I challenge the class to brainstorm a variety of different routes that the spider might take and to calculate the distances for each of these new routes. To help in their brainstorming, I suggest that they examine possible routes on a net drawing of the room instead of a three dimensional drawing. Some students might find it helpful to model the room with a manipulative which allows them to link polygons together (such as *Polydrons*) on which they could label the walls, floor and ceiling of the room as well as the position of the spider and the fly. This would allow students to see how the path might change depending on how they create the net of the room. You might challenge the students to find a net that results in the spider crossing three sides of the room, four sides of the room or even five sides of the room and to see how these different nets result in different distance paths. Eventually, students will find the solution of the shortest path. Below are four different nets that Dudeney showed in *The Canterbury Puzzles*.

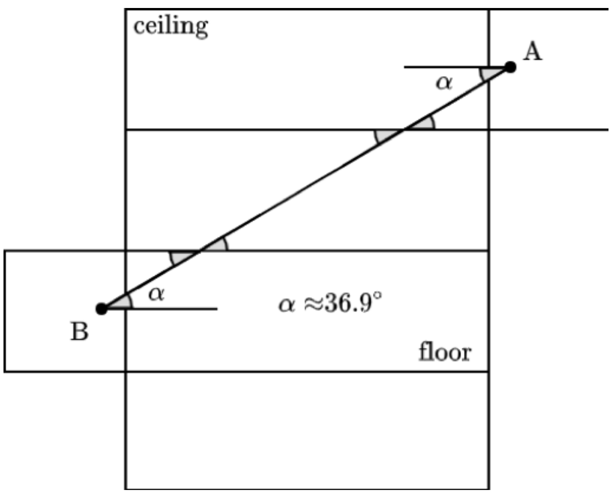


The distances for each of the solutions above can be found by applying the Pythagorean Theorem. Students at times have difficulty imaging the shortest path given in solution number 4. Returning to the pictures of the spider and the fly taped to the walls, a long piece of yarn is used to show the path of the spider along the sides of the classroom.

In his book *The Pythagorean Theorem: A 4,000-Year History*, Eli Maor describes how we could accurately trace the spider's path using trigonometry. In case 4 above, the spider's horizontal distance is 1 + 30 + 1 = 32 feet and the vertical distance travelled is 6 + 12 + 6 = 24 feet.

$$\tan\alpha = \frac{24}{32}$$
$$\arctan\frac{24}{32} = \alpha$$
$$\alpha \approx 36.9^\circ$$

The diagram below shows how the spider travels using this angle across the sides the room.



To imagine why this is the shortest path think about the shape of the room as a cylinder with hemispherical ends instead of a rectangular prism (like a hot dog instead of a block of wood). Imagine the piece of yarn wrapping around this shape from the spider to the fly's position. Now imagine if this "hot dog" shape slowly changed shape, or "deflated", until it was the rectangular prism. The curving path from the cylinder would now be the angled path of the spider around the classroom.

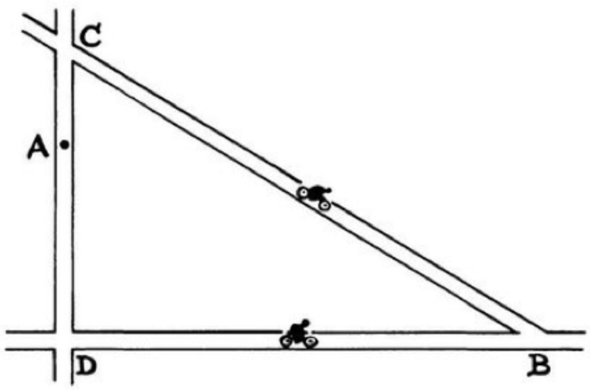
There are several extensions to this problem that could be explored:

- How do the dimensions of the room affect the shortest path that the spider takes? Would the path from the problem we solved be the shortest path for a room with any dimensions?
- How does the spider's height on the wall affect this problem? For which starting heights, h, would there be a different shortest route?
- Investigate the study of geodesics. How do geodesics apply to this problem?

A Follow Up Problem – The Russian Motorcyclists

The following is another problem from Henry Ernest Dudeney which was published in *The Strand Magazine*, Volume 53 (1917).

> Two Army motorcyclists, on the road at Adjbkmlprzll, wish to go to Brczrtwxy, which, for the sake of brevity, are marked in the accompanying map as A and B. Now, Pipipoff said: "I shall go to D, which is six miles, and then take the straight road to B, another fifteen miles." But Sliponsky thought he would try the upper road by way of C. Curiously enough, they found on reference to their cyclometers that the distance either way was exactly the same. This being so, they ought to have been able easily to answer the General's simple question, "How far is it from A to C?" It can be done in the head in a few moments, if you only know how. Can the reader state correctly the distance?



There are several ways to solve this problem with a bit of algebra and the application of the Pythagorean Theorem. Dudeney cryptically states that, "It can be done in the head in a few moments, if you only know how." Can you deduce the clever solution method that Dudeney is referring to?

Erick Lee is a Mathematics Support Consultant for the Halifax Regional Centre for Education in Dartmouth, NS. Erick blogs at `https://pbbmath.weebly.com/` *and can be reached via email at* `elee@hrce.ca` *and on Twitter at* `@TheErickLee`