MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(8), p. 447–449.

MA36. Let A and B be sets with the property that there are exactly 144 sets which are subsets of at least one of A or B. How many elements does the union of A and B have?

Originally problem 13 of "A Mathematical Orchard: Problems and Solutions" by Mark I. Krusemeyer, George T. Gilbert, and Loren C. Larson.

We received 5 submissions, of which 3 were correct and complete. We present the solution by Digby Smith, slightly modified by the editor.

Let M be the number of elements in A and N the number of elements in B; without loss of generality, assume $N \leq M$. Let K be the number of elements in the intersection of A and B, and note $0 \leq K \leq N$.

The number of subsets (including the empty set) of a set with n distinct elements is 2^n, so the number of sets which are subsets of at least one of A and B is $2^M + 2^N - 2^K$. We have

$$144 = 2^M + 2^N - 2^K < 2^M + 2^M = 2^{M+1}$$

which gives us $8 \leq M + 1$, so $7 \leq M$. On the other hand,

$$144 = 2^M + 2^N - 2^K \geq 2^M,$$

so $7 \geq M$, and we conclude that $M = 7$.

Hence there are $144 - 128 = 16$ nonempty subsets which are subsets of B but not of A. There are $2^N - 2^K$ sets which are subsets of B but not of A, so we must have $N > K$ and also

$$2^K(2^{(N-K)} - 1) = 2^4.$$

If $N - K > 1$ then $2^{(N-K)} - 1$ is an odd number greater than 1, which cannot divide 2^4. Thus we must have $K = 4$ and $N = 5$. Hence the number of elements in the union of A and B is $M + N - K = 8$.

MA37. Both 4 and 52 can be expressed as the sum of two squares as well as exceeding another square by 3:

$$4 = 0^2 + 2^2 \quad \text{and} \quad 4 - 3 = 1^2,$$
$$52 = 4^2 + 6^2 \quad \text{and} \quad 52 - 3 = 7^2.$$

Show that there are an infinite number of such numbers that have these two characteristics.

Originally problem 123 of "A Mathematical Orchard: Problems and Solutions" by Mark I. Krusemeyer, George T. Gilbert, and Loren C. Larson.

We received 7 submissions, of which 6 were correct and complete. We present the solution by Anita Hessami Pilehrood.

We need to show that there is a k such that $k = a^2 + b^2$ and $k = c^2 + 3$ for nonnegative integers a, b, c. Let $a = 2n$, $b = 2n^2 - 2$, and $c = 2n^2 - 1$ for a positive integer n. Then

$$a^2 + b^2 = 4n^2 + 4n^4 + 4 - 8n^2 = 4n^4 - 4n^2 + 4$$

and

$$c^2 + 3 = 4n^4 + 1 - 4n^2 + 3 = 4n^4 - 4n^2 + 4.$$

Therefore $a^2 + b^2 = c^2 + 3$ and hence there is an infinite sequence of numbers $k = 4(n^4 - n^2 + 1)$, $n = 1, 2, 3, \ldots$, that have these two characteristics.

MA38. Consider a 12×12 chessboard consisting of 144 1×1 squares. If three of the four corner squares are removed, can the remaining area be covered by placing 47 1×3 tiles?

Originally problem 33 of "A Mathematical Orchard: Problems and Solutions" by Mark I. Krusemeyer, George T. Gilbert, and Loren C. Larson.

We received 4 solutions, of which 3 were complete and correct. We present the solution by the Missouri State University Problem Solving Group.

Number the squares of the (intact) chessboard as shown in the figure.

3	1	2	3	1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2	3	1	2	3
3	1	2	3	1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2	3	1	2	3
3	1	2	3	1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2	3	1	2	3
3	1	2	3	1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2	3	1	2	3

Each number appears 48 times and any 1×3 tile will cover exactly one of each number, regardless of its position or orientation. Therefore, in order to tile the board with three corner squares removed, we must remove one of each square with a given number. In particular, we cannot remove the upper-left and lower-right corner squares, which are both labeled 3. Similarly, rotating the numbering scheme 90°, we conclude that we cannot remove the lower-left and upper-right corner squares. But removing any three corner squares will result in two diagonally opposite corners being removed and this gives us a contradiction.

MA39. Point E is selected on side AB of triangle ABC in such a way that $AE : EB = 1 : 3$ and point D is selected on side BC so that $CD : DB = 1 : 2$. The point of intersection of AD and CE is F. Determine the value of $\frac{EF}{FC} + \frac{AF}{FD}$.

Originally MAA Problem Book II (1961–1965), Question 37, 1965 examination.

We received 8 submissions, all correct. We present the solution provided by Anita Hessami Pilehrood.

Let the area of ΔCFD be y. Then $[DFB] = 2y$ since $\dfrac{DB}{DC} = 2$ and ΔDFB and ΔCFD share the height dropped from vertex F. Similarly, let the area of ΔAFE be x. Then the area of ΔEFB equals $3x$ since $\dfrac{EB}{AE} = 3$ and both triangles have a common height dropped from vertex F.

Now let's consider ΔBCE and ΔECA. Since $\dfrac{BE}{AE} = 3$ and these two triangles have a common height from vertex C, we have $\dfrac{[BCE]}{[ECA]} = 3$ which implies

$$\frac{y + 2y + 3x}{[CFA] + x} = 3,$$

and thus $[CFA] = y$.

Similarly in ΔBAD and ΔCAD, we have $\dfrac{BD}{DC} = 2$ and a common height from vertex A. Thus,

$$\frac{[BAD]}{[CAD]} = 2 \quad \text{implies} \quad \frac{4x + 2y}{2y} = 2, \quad \text{so} \quad \frac{x}{y} = \frac{1}{2}.$$

We have

$$\frac{EF}{FC} = \frac{[EAF]}{[FAC]} = \frac{x}{y} = \frac{1}{2},$$
$$\frac{AF}{FD} = \frac{[ACF]}{[FCD]} = \frac{y}{y} = 1,$$

and therefore

$$\frac{EF}{FC} + \frac{AF}{FD} = \frac{1}{2} + 1 = \frac{3}{2}.$$

MA40. In racing over a given distance d at uniform speeds, A can beat B by 20 yards, B can beat C by 10 yards, and A can beat C by 28 yards. Determine the distance d in yards.

Originally MAA Problem Book II (1961–1965), Question 37, 1961 examination.

We received 7 submissions, all of which were complete and correct. We present the solution of Aaratrick Basu, lightly edited.

Let v_A be the speed of A, v_B be the speed of B, and v_C be the speed of C.

As per the problem, A beats B by 20 yards, i.e,

$$\frac{d}{v_A} = \frac{d-20}{v_B} \tag{1}$$

Similarly, we have

$$\frac{d}{v_B} = \frac{d-10}{v_C}, \tag{2}$$

$$\frac{d}{v_A} = \frac{d-28}{v_C}. \tag{3}$$

Then from (1) and (3) we get

$$\frac{d-20}{v_B} = \frac{d-28}{v_C} \implies \frac{v_B}{v_C} = \frac{d-20}{d-28}.$$

With (2) this becomes

$$\frac{d}{d-10} = \frac{d-20}{d-28} \implies d^2 - 30d + 200 = d^2 - 28d,$$

which reveals $d = 100$. Hence, we have that A, B, C were racing over $d = 100$ yards.