FOCUS ON...

No. 40

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Inequalities *via* auxiliary functions (I)

Introduction

In attempts at proving an inequality, a resort to the study of an auxiliary function often arises naturally. Most of the time, choosing an appropriate function and using calculus to obtain its variations lead to a solution. The goal of this number is to illustrate the method through various examples.

From their very definition, convex functions are connected to inequalities and consequently regularly appear in the treatment of inequalities. In this first part, we will leave them aside, devoting our next number to their use.

A series of simple examples

We start with some cases when the auxiliary function is readily deduced from the proposed inequality itself.

Our first example is problem **2970** [2004 : 368, 371 ; 2005 : 414]:

> If m and n are positive integers such that $m \geq n$, and if $a, b, c > 0$, prove that
>
> $$\frac{a^m}{b^m + c^m} + \frac{b^m}{c^m + a^m} + \frac{c^m}{a^m + b^m} \geq \frac{a^n}{b^n + c^n} + \frac{b^n}{c^n + a^n} + \frac{c^n}{a^n + b^n}\,.$$

It is quite natural to introduce the function f defined on $[0, \infty)$ by

$$f(x) = \frac{a^x}{b^x + c^x} + \frac{b^x}{c^x + a^x} + \frac{c^x}{a^x + b^x}.$$

The key is that we may suppose $a \geq b \geq c$ (since $f(x)$ remains unchanged when a, b, c are permuted). The derivative of f is easily calculated:

$$f'(x) = \sum_{cyclic} \frac{a^x}{(b^x + c^x)^2}\left(b^x \ln\left(\frac{a}{b}\right) + c^x \ln\left(\frac{a}{c}\right)\right)$$

$$= a^x b^x \ln\left(\frac{a}{b}\right)\left(\frac{1}{(b^x + c^x)^2} - \frac{1}{(c^x + a^x)^2}\right) + b^x c^x \ln\left(\frac{b}{c}\right)\left(\frac{1}{(c^x + a^x)^2} - \frac{1}{(a^x + b^x)^2}\right)$$

$$+ c^x a^x \ln\left(\frac{a}{c}\right)\left(\frac{1}{(b^x + c^x)^2} - \frac{1}{(a^x + b^x)^2}\right).$$

Since $a \geq b \geq c$, the numbers $\ln\left(\frac{a}{b}\right), \ln\left(\frac{b}{c}\right), \ln\left(\frac{a}{c}\right)$ are nonnegative and for $x \geq 0$ we have $(b^x + c^x)^2 \leq (c^x + a^x)^2 \leq (a^x + b^x)^2$, whence $f'(x) \geq 0$ (the three summands above are nonnegative). Thus, f is nondecreasing on $[0, \infty)$ and $f(u) \geq f(v)$ whenever $u \geq v \geq 0$, which is more general than the required result.

The auxiliary function is also chosen at once in the next example, problem **3889** [2013 : 413 ; 2014 : 404]:

> Prove that
> $$e^{\pi} > \left(\frac{e^2+\pi^2}{2e}\right)^e.$$

First we take logarithms, transforming the given inequality into

$$\frac{\pi}{e} > \ln\left(\frac{e}{2}\right) + \ln\left(1+\frac{\pi^2}{e^2}\right), \tag{1}$$

and define the auxiliary function f by $f(x) = x - \ln(1+x^2)$. Its derivative $f'(x) = \frac{(x-1)^2}{x^2+1}$ is positive for $x \in (1, \infty)$, hence f is increasing on $[1, \infty)$. As a result, $f\left(\frac{\pi}{e}\right) > f(1) = 1 - \ln(2)$ and (1) follows.

In our last example, problem **2933** [2004 : 173 ; 2005 : 186], the auxiliary function is less obvious:

> Prove, without the use of a calculator, that $\sin(40^\circ) < \sqrt{\frac{3}{7}}$.

Here the key observation is that $\sin(3 \times 40^\circ)$ is well-known. Recalling the formula $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, we are led to introduce the function f defined by $f(x) = 3x - 4x^3$. From the derivative $f'(x) = 3(1-2x)(1+2x)$, we see that f is decreasing on the interval on $(1/2, 1]$.

Both $\sin(40^\circ)$ and $\sqrt{\frac{3}{7}}$ lie in this interval and, in addition to $\sin(3 \times 40^\circ) = \sin(120^\circ) = \frac{\sqrt{3}}{2}$, a short calculation gives $f(\sqrt{\frac{3}{7}}) = \frac{9}{7}\sqrt{\frac{3}{7}}$. From $\frac{7}{4} > \frac{81}{49}$, we deduce that $\frac{3}{4} > \frac{3}{7} \cdot \frac{81}{49}$, hence $\frac{\sqrt{3}}{2} > \frac{9}{7}\sqrt{\frac{3}{7}}$. Thus, $f(\sin(40^\circ)) > f\left(\sqrt{\frac{3}{7}}\right)$ and $\sin(40^\circ) < \sqrt{\frac{3}{7}}$ follows.

Auxiliary functions in succession

The resort to two or more auxiliary functions frequently occurs, for instance when another study of function is needed to obtain the sign of a derivative. A good example is extracted from problem **3228** [2007 : 169, 172 ; 2008 : 178]:

> For $x \in (0, \frac{\pi}{2})$, prove that
> $$\frac{x}{\sin x} \leq \frac{\pi}{2+\cos x}.$$

The inequality is equivalent to $\psi(x) \geq 0$ where $\psi(x) = \pi \sin x - 2x - x\cos x$. We calculate

$$\psi'(x) = (\pi - 1)\cos x + x\sin x - 2,$$
$$\psi''(x) = x\cos x - (\pi - 2)\sin x,$$
$$\psi'''(x) = -x\sin x - (\pi - 3)\cos x.$$

Since $\psi'''(x) < 0$ for $x \in [0, \frac{\pi}{2}]$, the function ψ'' is decreasing. Remarking that $\psi''(0) = 0$, we deduce that $\psi''(x) < 0$ for $x \in (0, \frac{\pi}{2}]$ and ψ' is decreasing as well. Since $\psi'(0) > 0$, $\psi'(\frac{\pi}{2}) < 0$ we have $\psi'(\alpha) = 0$ for some unique $\alpha \in (0, \frac{\pi}{2})$.

It follows that ψ is increasing on $(0, \alpha)$ and decreasing on $(\alpha, \frac{\pi}{2})$. Observing that $\psi(0) = \psi(\frac{\pi}{2}) = 0$, we may conclude that $\psi(x) > 0$ for $x \in (0, \frac{\pi}{2})$.

Our next example is problem **4061** [2015 : 302, 303 ; 2016 : 318]. We offer a variant of solution making use of two independent auxiliary functions.

> Let ABC be a non-obtuse triangle none of whose angles are less than $\frac{\pi}{4}$. Find the minimum value of $\sin A \sin B \sin C$.

We begin by obtaining inequalities about two auxiliary functions:

(a) For $x \in [\frac{\pi}{4}, \frac{\pi}{3}]$, let $f(x) = \sin^2 x \sin 2x$. Then $f(x) \geq \frac{1}{2}$.

Proof. The derivative f' satisfies

$$f'(x) = 2\sin^2 x(1 + 2\cos 2x) \geq 0$$

(since $\frac{\pi}{2} \leq 2x \leq \frac{2\pi}{3}$), hence $f(x) \geq f(\frac{\pi}{4}) = \frac{1}{2}$. □

(b) Let θ be a fixed real number in $[\frac{\pi}{4}, \frac{\pi}{3}]$. For $x \in [\theta, \frac{\pi-\theta}{2}]$, let

$$g_\theta(x) = \sin x \sin(x + \theta).$$

Then $g_\theta(x) \geq \sin\theta \sin 2\theta$.

Proof. Here $g'_\theta(x) = \sin(2x + \theta) \geq 0$ (since $0 < 3\theta \leq 2x + \theta \leq \pi$), hence

$$g_\theta(x) \geq g_\theta(\theta) = \sin\theta \sin 2\theta.$$

□

Turning to the problem, we may suppose that $C \leq B \leq A$. Then, $\frac{\pi}{4} \leq C \leq \frac{\pi}{3}$ (note that $3C \leq A + B + C = \pi$) and $B \leq \pi - B - C$ so that $C \leq B \leq \frac{\pi - C}{2}$.

Now, applying successively (b) and (a), we obtain

$$\begin{aligned}\sin A \sin B \sin C &= \sin C \sin B \sin(B + C)\\ &= \sin C \cdot g_C(B)\\ &\geq \sin C \cdot \sin C \sin 2C\\ &= f(C) \geq \frac{1}{2}.\end{aligned}$$

In addition, $\sin A \sin B \sin C = \frac{1}{2}$ if $A = \frac{\pi}{2}$, $B = C = \frac{\pi}{4}$. Thus, the desired minimum value is $\frac{1}{2}$.

To conclude, we consider problem **4267** [2017 : 303, 305 ; 2018 : 311]. We propose a solution, which, if longer than the featured one, may show to the beginner how to deal with a difficult inequality in a natural way.

> Let a, b, c and d be real numbers such that $0 < a, b, c \leq 1$ and $abcd = 1$. Prove that
> $$5(a+b+c+d) + \frac{4}{abc+abd+acd+bcd} \geq 21.$$

Since $abcd = 1$, the inequality is equivalent to $L \geq 21$ where

$$L = 5\left(a+b+c+\frac{1}{abc}\right) + \frac{4}{abc + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}}.$$

To prove the inequality $L \geq 21$, we use a chain of auxiliary functions.

Let a, b be fixed in $(0, 1]$ and $f(x) = 5\left(a+b+x+\frac{1}{abx}\right) + \frac{4}{abx+\frac{1}{a}+\frac{1}{b}+\frac{1}{x}}$ so that $L = f(c)$. We calculate the derivative of f in $(0, 1]$:

$$f'(x) = \frac{1-abx^2}{x^2}\left(\frac{4}{\left(abx+\frac{1}{a}+\frac{1}{b}+\frac{1}{x}\right)^2} - \frac{5}{ab}\right).$$

Since $0 < a, b, x \leq 1$ we have $1 - abx^2 \geq 0$ and on the other hand $\frac{1}{a} + \frac{1}{b} + \frac{1}{x} \geq 3$, hence

$$\frac{4}{\left(abx+\frac{1}{a}+\frac{1}{b}+\frac{1}{x}\right)^2} \leq \frac{4}{9}$$

while $\frac{5}{ab} \geq 5$. Therefore $f'(x) < 0$ for $x \in (0, 1]$. It follows that f is decreasing on $(0, 1]$ and so

$$f(c) \geq f(1) = 5\left(a+b+1+\frac{1}{ab}\right) + \frac{4}{ab+\frac{1}{a}+\frac{1}{b}+1} = g(b)$$

where $g(x) = 5\left(a+x+1+\frac{1}{ax}\right) + \frac{4}{ax+\frac{1}{a}+\frac{1}{x}+1}$.
Similarly,

$$g'(x) = \frac{1-ax^2}{x^2}\left(\frac{4}{\left(ax+\frac{1}{a}+1+\frac{1}{x}\right)^2} - \frac{5}{a}\right)$$

is negative on $(0, 1]$ and so

$$g(b) \geq g(1) = 5\left(2+a+\frac{1}{a}\right) + \frac{4}{2+a+\frac{1}{a}} = h\left(2+a+\frac{1}{a}\right)$$

where $h(x) = 5x + \frac{4}{x}$. Since $2 + a + \frac{1}{a} \geq 2 + 2 = 4$, we study h on the interval $[4, \infty)$. On this interval, $h'(x) = 5 - \frac{4}{x^2} > 0$ so that h is increasing. Consequently

$$h\left(2+a+\frac{1}{a}\right) \geq h(4) = 21.$$

In conclusion, we have

$$L = f(c) \geq g(b) \geq h\left(2 + a + \frac{1}{a}\right) \geq 21$$

and the required inequality follows.

About the choice of an auxiliary function

To avoid a complicated study, it is sometimes better to delay the introduction of an auxiliary function. We give two examples of such situations. First, here is a variant of solution to problem **3908** [2014 : 29, 31 ; 2015 : 39]:

> Prove that $\dfrac{(n-1)^{2n-2}}{(n-2)^{n-2}} < n^n$ for each integer $n \geq 3$.

The function $x \mapsto x^x - \frac{(x-1)^{2x-2}}{(x-2)^{x-2}}$ does not seem a very good choice! We rewrite the inequality in a more convenient form:

$$\left(1 + \frac{1}{n(n-2)}\right)^n < \left(1 + \frac{1}{n-2}\right)^2.$$

But once again, a function like $x \mapsto \left(1 + \frac{1}{x}\right)^2 - \left(1 + \frac{1}{x(x+2)}\right)^x$ would not lead to a nice study! However, recalling that for any positive real number x and any positive integer n,

$$\left(1 + \frac{x}{n}\right)^n < e^x$$

we see that it is sufficient to prove that

$$\left(1 + \frac{1}{n-2}\right)^2 > e^{1/(n-2)}$$

for $n \geq 3$. At that stage we can efficiently consider $f(x) = (1+x)^2 - e^x$ for $x \in [0,1]$. A quick study of the derivative $f'(x) = 2(1+x) - e^x$ shows that f' is increasing on $[0, \ln 2]$ and decreasing on $[\ln 2, 1]$. Since $f'(0) = 1, f'(1) = 4 - e > 0$, it follows that $f'(x) > 0$ for all $x \in [0,1]$. Therefore f is increasing on $[0,1]$ and $f(x) > f(0) = 0$ whenever $x \in (0,1]$ and the desired inequality follows.

A similar difficulty is to be found in problem **3929** [2014 : 122,124 ; 2015 : 135]:

> Show that for all $0 < x < \pi/2$, the following inequality holds:
>
> $$\left(1 + \frac{1}{\sin x}\right)\left(1 + \frac{1}{\cos x}\right) \geq 5\left[1 + x^4\left(\frac{\pi}{2} - x\right)^4\right].$$

The inequality is $f(x) \geq 4 + 5x^4\left(\frac{\pi}{2} - x\right)^4$ where

$$f(x) = \frac{1}{\sin x} + \frac{1}{\cos x} + \frac{2}{\sin 2x}.$$

Again, it is better to remark that for $0 < x < \pi/2$ we have

$$0 < x\left(\frac{\pi}{2} - x\right) \leq \left(\frac{x + \frac{\pi}{2} - x}{2}\right)^2 = \frac{\pi^2}{16},$$

hence

$$4 + 5x^4\left(\frac{\pi}{2} - x\right)^4 \leq 4 + 5\left(\frac{\pi^2}{16}\right)^4$$

and therefore it is enough to prove that $f(x) \geq 4 + 5\left(\frac{\pi^2}{16}\right)^4$. Now, we readily obtain that $f'(x)$ has the same sign as $g(x) = \sin^3 x - \cos^3 x - \cos 2x$. A quick study of g then will show that $f(x) \geq f(\pi/4) = 2 + 2\sqrt{2}$ [details are left to the reader] and the conclusion follows from $2 + 2\sqrt{2} > 4 + 5\left(\frac{\pi^2}{16}\right)^4$.

As usual, we end this number with a couple of exercises.

Exercises

1. Let $n \in \mathbb{N}$ and let

$$\Delta(x_1, x_2, \ldots, x_n) = \sum_{i=1}^n x_i - \prod_{i=1}^n x_i.$$

If $a_1, a_2, \ldots, a_n \in (0, 1]$ prove that

$$\Delta(a_1, a_2, \ldots, a_n) \geq \Delta\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right).$$

2. For $y \in (0, 1]$, let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = y^x + x^y - 1$ and $g : (0, 1] \to \mathbb{R}$ be defined by $g(x) = f(x) - \frac{x}{y} \cdot f'(x)$. From the study of g deduce that $f(x) > 0$ for $x \in (0, 1]$.

3. (inspired by problem **1061** of *the College Mathematics Journal*) Let m be an integer with $m \geq 2$ and r a real number in $[1, \infty)$. Prove that

$$\left(\frac{1 + r^m}{1 + r^{m-1}}\right)^{m+1} \geq \frac{1 + r^{m+1}}{2}.$$

[Hint: determine the sign of $u(x) = (m - 1)(1 + x^{m+1}) - x(1 + x^{m-1})$ for $x \geq 1$.]