SOLUTIONS

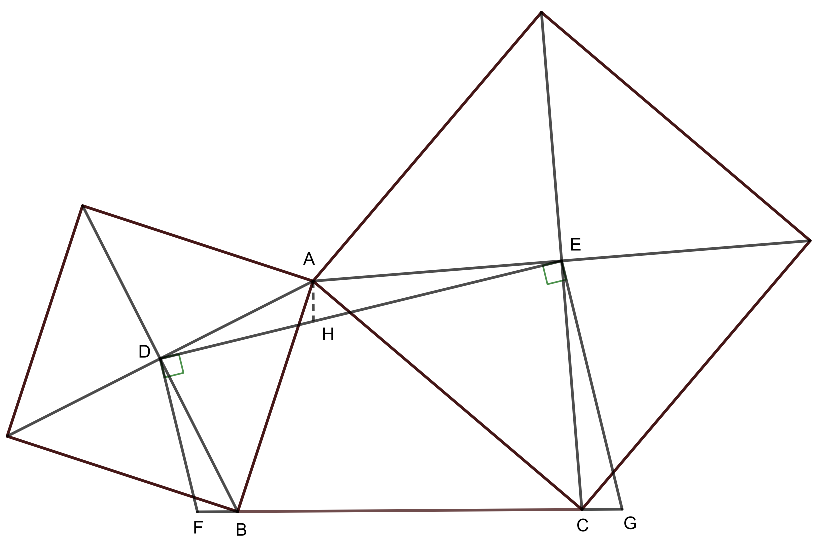
No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2019: 45(1), p. 33–37; 45(2), p. 85–89.

4401. *Proposed by Ruben Dario and Leonard Giugiuc.*

Let D and E be the centres of squares erected externally on the sides AB and AC, respectively, of an arbitrary triangle ABC, and define F and G to be the intersections of the line BC with lines perpendicular to ED at D and at E. Prove that the resulting segments BF and CG are congruent.

We received 11 solutions. We present the solution by Madhav Modak.



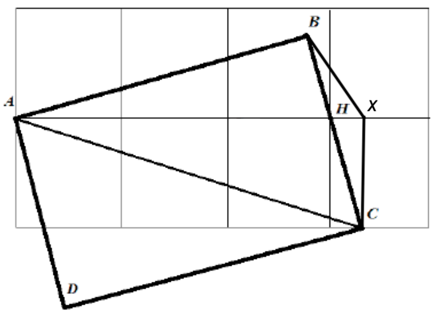
Let H be the point on DE such that $DH = DF$. Since D is the centre of the square on side AB we have $DB \perp DA$ and $DB = DA$. Further, we are given that $DF \perp DH$, and so $\angle ADH = 90^\circ - \angle HDB = \angle BDF$. It follows that $\triangle ADH$ and $\triangle BDF$ are congruent. Hence $FB = HA$. Moreover,

$$\angle AHE = 180^\circ - \angle AHD = 180^\circ - \angle BFD,$$

and using the fact that $DF||EG$ we conclude that $\angle AHE = \angle ECG$. As before, since $AE \perp EC$ and $HE \perp EG$, we get that $\angle AEH = \angle CEG$. Using the fact that $AE = EC$, we get that $\triangle AEH$ and $\triangle CEG$ are congruent, so $AH = CG$. Therefore, $FB = CG$, as desired.

4402. *Proposed by Peter Y. Woo.*

Consider a rectangular carpet $ABCD$ lying on top of floor tiled with 8 square tiles with side length of 1 foot each (as shown in the diagram).



Suppose AH bisects $\angle BAC$. Express $\tan \angle BAH$ as the sum of a rational number and the square root of a rational number.

We received 15 submissions, all correct. We present the solution by Jirapat Kaewkam, enhanced by the editor.

Let CX be perpendicular to the horizontal line l extended from AH with X being on l. Connect CX (see figure). Since $\angle ABC = \angle AXC = 90°$, we see that A, B, X, C are concyclic so $\angle XCH = \angle BAX = \angle XAC$. Hence $\Delta XCH \sim \Delta XAC$ from which it follows that

$$\frac{AX}{CX} = \frac{CX}{HX} \quad \text{or} \quad (AX)(HX) = (CX)^2 = 1,$$

so $(HX + 3)(HX) = 1$. Solving

$$(HX)^2 + 3(HX) - 1 = 0,$$

we then obtain

$$\tan(\angle BAH) = \tan(\angle XCH) = \frac{HX}{CX} = HX = \frac{-3 + \sqrt{13}}{2} = -\frac{3}{2} + \sqrt{\frac{13}{4}}.$$

4403. *Proposed by Michel Bataille.*

Let m be an integer with $m > 1$. Evaluate in closed form

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n+1}{k+1} \frac{k}{m+k}.$$

We received 8 submissions, all of which were correct and complete. We present two solutions.

Solution 1, by the proposer.

Let
$$S_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n+1}{k+1} \frac{k}{m+k}.$$
We show that
$$S_n = \frac{1}{m-1}\left(1 - \frac{n+1}{\binom{m+n}{m}}\right).$$
Using the well-known identity
$$\sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1}$$
and changing the order of summation, we obtain
$$S_n = \sum_{j=1}^{n} \sum_{k=1}^{j} (-1)^{k-1} \frac{k}{m+k} \binom{j}{k}.$$

Taking the relation $k\binom{j}{k} = j\binom{j-1}{k-1}$ into account yields
$$S_n = \sum_{j=1}^{n} j \sum_{k=1}^{j} (-1)^{k-1} \frac{1}{m+k} \binom{j-1}{k-1}. \tag{1}$$
Now we use the closed form
$$\sum_{k=1}^{j} (-1)^{k-1} \frac{1}{m+k} \binom{j-1}{k-1} = \frac{(j-1)!}{(m+1)(m+2)\cdots(m+j)}$$
which readily follows from the decomposition of $\frac{1}{(m+1)(m+2)\cdots(m+j)}$ into partial fractions. Back to (1), this leads to
$$S_n = \sum_{j=1}^{n} \frac{j!}{(m+1)(m+2)\cdots(m+j)} = m! \sum_{j=1}^{n} \frac{1}{(j+1)(j+2)\cdots(j+m)}. \tag{2}$$
But we have
$$\begin{aligned} &\frac{1}{(j+1)(j+2)\cdots(j+m)} \\ &= \frac{1}{m-1}\left(\frac{1}{(j+1)(j+2)\cdots(j+m-1)} - \frac{1}{(j+2)(j+3)\cdots(j+m)}\right). \end{aligned}$$
So that the last sum in (2) is telescopic and therefore
$$S_n = \frac{m!}{m-1}\left(\frac{1}{m!} - \frac{1}{(n+2)\cdots(n+m)}\right) = \frac{1}{m-1}\left(1 - \frac{n+1}{\binom{m+n}{m}}\right).$$

Solution 2, by Paul Bracken, Brian Bradie, Madhav Modak, CR Pranesachar, and Daniel Văcaru, all done independently.

We have

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1}\frac{k}{m+k}\\ &= \sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1} - m\sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1}\frac{1}{m+k}. \qquad (3)\end{aligned}$$

Using binomial expansion,

$$\begin{aligned}0 = (1-1)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k &= \sum_{k=-1}^{n}(-1)^{k+1}\binom{n+1}{k+1}\\ &= -n + \sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1}. \qquad (4)\end{aligned}$$

Again by binomial expansion

$$(1-x)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k x^k = 1-(n+1)x + \sum_{k=1}^{n}\binom{n+1}{k+1}(-1)^{k-1}x^{k+1}.$$

It follows that

$$x^{m-2}((1-x)^{n+1} - 1 + (n+1)x) = \sum_{k=1}^{n}\binom{n+1}{k+1}(-1)^{k-1}x^{m+k-1}.$$

Since $m > 1$, we have

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1}\frac{1}{m+k} &= \int_0^1 \sum_{k=1}^{n}\binom{n+1}{k+1}(-1)^{k-1}x^{m+k-1}\,dx\\ &= \int_0^1 x^{m-2}((1-x)^{n+1} - 1 + (n+1)x)\,dx.\end{aligned}$$

The integral $\int_0^1 x^{m-2}(1-x)^{n+1}\,dx$ is a beta function with the value $\frac{(m-2)!(n+1)!}{(m+n)!}$. Therefore the above integral has the value

$$\frac{(m-2)!(n+1)!}{(m+n)!} + \frac{n+1}{m} - \frac{1}{m-1}.$$

Combining this with the result in (4) into (3) we have

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1}\frac{k}{m+k} &= n - m\left(\frac{(m-2)!(n+1)!}{(m+n)!} + \frac{n+1}{m} - \frac{1}{m-1}\right)\\ &= \frac{1}{m-1}\left(1 - \frac{n+1}{\binom{m+n}{m}}\right).\end{aligned}$$

4404. *Proposed by Nguyen Viet Hung.*

Let x, y and z be integers such that $x > 0, z > 0$ and $x + y > 0$. Find all the solutions to the equation

$$x^4 + y^4 + (x + y)^4 = 2(z^2 + 40).$$

We received 11 submissions, 9 of which were correct and complete. We present the solution by Brian D. Beasley.

The given equation is equivalent to

$$(x^2 + xy + y^2)^2 = z^2 + 40.$$

Letting

$$n = x^2 + xy + y^2,$$

we note that $n > 0$ and

$$(n + z)(n - z) = 40.$$

Since $z > 0$, this implies

$$(n + z, n - z) = (40, 1),\ (20, 2),\ (10, 4),\ \text{or}\ (8, 5).$$

But

$$(n + z) + (n - z) = 2n$$

must be even, so $2n = 22$ or $2n = 14$, and hence $(n, z) = (11, 9)$ or $(n, z) = (7, 3)$.

If $n = x(x+y)+y^2 = 11$ with $x > 0$ and $x+y > 0$, then $y \in \{0, \pm1, \pm2, \pm3\}$. But none of these values for y will yield an integer value for x.

If $n = x(x + y) + y^2 = 7$ with $x > 0$ and $x + y > 0$, then $y \in \{0, \pm1, \pm2\}$. Four of these five values for y yield a positive integer value for x. Thus there are four solutions for (x, y, z) to the original equation:

$$(1, 2, 3),\ (2, 1, 3),\ (3, -1, 3),\ (3, -2, 3).$$

4405. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Let ABC be a triangle and let K be a point inside ABC. Suppose that BK intersects AC in F and CK intersects AB in E. Let M be the midpoint of BE, N be the midpoint of CF and suppose that MN intersects BK at P. Show that the midpoints of AF, EK and MP are collinear.

All 7 of the submissions we received were correct, but two of them relied on a computer. We present the solution by Andrea Fanchini.

We use barycentric coordinates with respect to triangle ABC. Working backwards, for the points E and F to be

$$E = CK \cap AB = (m, 1 - m, 0), \qquad F = BK \cap AC = (1 - n, 0, n)$$

(where m, n are parameters with $0 < m, n < 1$), we must have

$$BK : nx - (1-n)z = 0, \quad CK : (1-m)x - my = 0,$$

and, finally,

$$K\left(m(1-n) : (1-m)(1-n) : mn\right).$$

The midpoints of BE and of CF are then

$$M = (m : 2-m : 0), \qquad N = (1-n : 0 : 1+n).$$

The line $MN : (2-m)(1+n)x - m(1+n)y - (1-n)(2-m)z = 0$ intersects BK at P, so that

$$P = \big(m(1-n^2) : (2-m)(1-n) : mn(1+n)\big).$$

Finally, the midpoints of AF, EK and MP are

$$M_{AF} = (2-n : 0 : n),$$

$$M_{EK} = (m(mn-2n+2) : (1-m)(mn-2n+2) : mn),$$

and

$$M_{MP} = (m(mn-n^2-n+2) : (2-m)(mn-2n+2) : mn(1+n)).$$

Because

$$\begin{vmatrix} 2-n & 0 & n \\ m(mn-2n+2) & (1-m)(mn-2n+2) & mn \\ m(mn-n^2-n+2) & (2-m)(mn-2n+2) & mn(1+n) \end{vmatrix} = 0,$$

it follows that these midpoints are collinear.

Editor's comments. Note that there is no need to restrict K to the interior of the triangle: it could any point in the plane except B or C. In other words, we can allow m and n to be any real numbers except $m \neq 0$ and $n \neq 1$, and the result continues to hold.

It is interesting to compare our problem with an extended version of Hjelmslev's theorem:

> When all the points P on one line are related by a similarity to all the points P' on a second line, then the points X dividing the segments PP' in a fixed ratio $PX : XP'$ are distinct and collinear or else they all coincide.

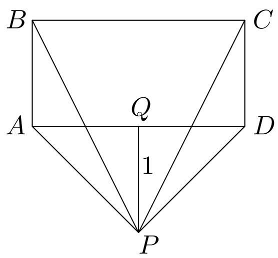
See, for example, F. G.-M., *Exercices de géométrie*, 4th ed., Theorem 1146d, page 473. Does any reader see an easy direct proof that there exists a similarity that takes the points A, E, M to the points F, K, P ?

4406. *Proposed by Bill Sands.*

Four trees are situated at the corners of a rectangle. You are standing outside the rectangle, the nearest point of the rectangle being the midpoint of one of its sides, 1 metre away from you. To you in this position, the four trees appear to be equally spaced apart.

a) Find the side lengths of the rectangle, assuming that they are positive integers.

b) Suppose that the rectangle is a square. Find the length of its side.



We received 8 correct solutions. We present two different approaches.

Solution 1, by Roy Barbara and C.R. Pranesachar (independently).

Let $\theta = \angle QPC$, so that

$$2\theta = \angle APB = \angle BPC = \angle CPD$$

and $3\theta = \angle QPD$. Let the length of BC be u and the length of CD be $v-1$. Then $\tan\theta = u/(2v)$ and

$$\frac{u}{2} = \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \frac{12uv^2 - u^3}{2(4v^3 - 3u^2v)}.$$

Hence $4v^3 - 3u^2v = 12v^2 - u^2$, whereupon $u^2(3v-1) = 4v^2(v-3)$.

(a) Let u and v be positive integers. Since $3v-1$ and v^2 are coprime, $3v-1$ must divide $4(v-3)$. Because

$$2(3v-1) - 4(v-3) = 2v + 10 > 0,$$

we must have $3v - 1 = 4(v-3)$ and $v = 11$. Hence $u = v = 11$ and the respective lengths of CD and BC are 10 and 11.

An alternative argument begins by rewriting the foregoing equation as

$$27u^2 = 36v^2 - 96v - 32 - \frac{32}{3v-1}.$$

The right side is positive and $0 \neq 4v^2(v-3)$, so that v exceeds 3. Since $3v-1$ divides 32, the only possibility is $v = 11$.

(b) If $ABCD$ is square, then $v = u + 1$ and so

$$\begin{aligned} 0 &= 4(u+1)^2(u-2) - u^2(3u+2) \\ &= u^3 - 2u^2 - 12u - 8 \\ &= (u+2)(u^2 - 4u - 4). \end{aligned}$$

Hence the sidelength of the square is $u = 2(1 + \sqrt{2})$.

Solution 2, by Daniel Vacaru.

Let the respective lengths of AP, AB and AD be s, t, u. Let $\theta = \angle BPQ$. Since $PQ \| AB$, $\angle ABP = \theta$. Also $\angle APB = 2\theta$ and $\angle APQ = 3\theta$. By the Law of Sines, $\sin 2\theta / t = \sin \theta / s$, whereupon $\cos \theta = t/(2s)$.

Since

$$\frac{1}{s} = \cos 3\theta = 4\cos^3\theta - 3\cos\theta = \frac{t^3 - 3ts^2}{2s^3},$$

then $(3t+2)s^2 = t^3$. Setting $s^2 = 1 + (u^2/4)$ yields that

$$(3t+2)(4+u^2) = 4t^3 \qquad \Longrightarrow \qquad (3t+2)u^2 = 4t^3 - 12t - 8.$$

(a) Let t and u be positive integers. Rewrite the equation as

$$27u^2 = 36t^2 - 24t - 92 - \frac{32}{3t+2}.$$

Since $3t+2$ divides 32, either $t = 2$ or $t = 10$. The first option leads to $(t, u) = (2, 0)$ which is inadmissible, and the second to $(t, u) = (10, 11)$.

(b) When $u = t$, the equation becomes

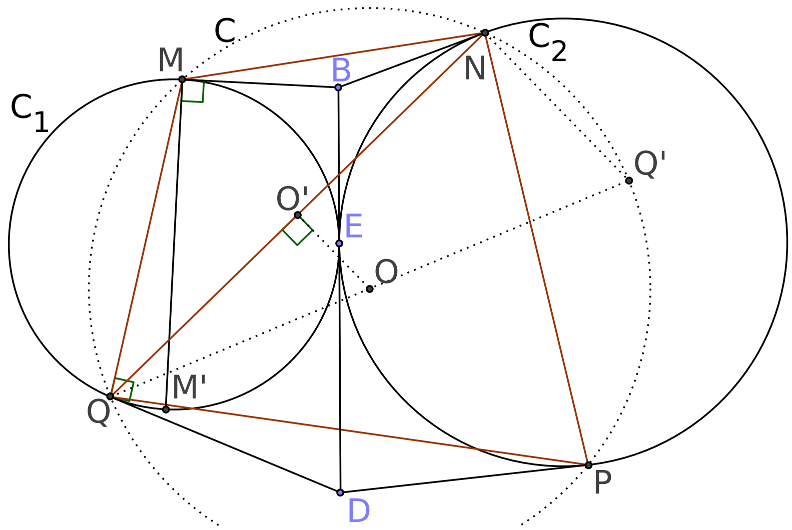
$$0 = t^3 - 2t^2 - 12t - 8 = (t+2)(t^2 - 4t - 4),$$

and the value $2(1 + \sqrt{2})$ for the sidelength of the square.

4407. *Proposed by Mihaela Berindeanu.*

Circle C_1 lies outside circle C_2 and is tangent to it at E. Take arbitrary points B and D different from E on the common tangent line. Let the second tangent from B to C_1 touch it at M and to C_2 touch it at N, while the second tangents from D to those circles touch them at Q and P, respectively. If the orthocenters of the triangles MNQ and PNQ are H_1 and H_2, prove that $\overrightarrow{H_1H_2} = \overrightarrow{MP}$.

We received 6 submissions of which 5 were correct. We present the solution by Oliver Geupel.



Let point M' be the antipode of M on C_1. We use directed angles ($\measuredangle$) modulo 180°. In $\Delta QM'M$ and its circumcircle we have

$$\measuredangle QMM' = \measuredangle MQM' + \measuredangle QM'M = 90^\circ + \measuredangle QEM.$$

Since the points E, M, and N all lie on a common circle centered at B, we have that $\measuredangle MEN = \frac{1}{2}\measuredangle MBN$, so

$$\measuredangle BMN = 90^\circ + \measuredangle MEN.$$

Hence,

$$\begin{aligned} \measuredangle QMN &= \measuredangle QMM' + \measuredangle M'MB + \measuredangle BMN \\ &= 90^\circ + \measuredangle QEM + 90^\circ + 90^\circ + \measuredangle MEN \\ &= 90^\circ + \measuredangle QEN. \end{aligned}$$

Similarly (replacing Q, M, N, B by N, P, Q, D, respectively),

$$\measuredangle QPN = 90^\circ + \measuredangle QEN.$$

Thus, $\measuredangle QMN = \measuredangle QPN$, which implies that the points M, N P, and Q all lie on a common circle, say, C with center O.

Let O' be the midpoint of the segment NQ. Let point Q' be the antipode of Q on C. The segment $O'O$ joins the midpoints of two sides of the right triangle NQQ'. Hence $2\overrightarrow{O'O} = \overrightarrow{NQ'}$. On the other hand, the lines MQ' and H_1N are both perpendicular to MQ, and the lines $Q'N$ and MH_1 are both perpendicular to NQ. Hence, the quadrilateral $H_1MQ'N$ is a parallelogram, from which we deduce $\overrightarrow{NQ'} = \overrightarrow{H_1M}$. It follows that $\overrightarrow{H_1M} = 2\overrightarrow{O'O}$. Analogously, $\overrightarrow{H_2P} = 2\overrightarrow{O'O}$. We conclude that

$$\overrightarrow{H_1M} = \overrightarrow{H_2P}.$$

Consequently, the quadrilateral H_1MPH_2 is a parallelogram, which proves the desired result.

Editor's comments. The person who submitted the faulty solution misread the problem, labeling the figure so that N and Q are on C_2. Interestingly, the result $\overrightarrow{H_1H_2} = \overrightarrow{MP}$ continues to hold in the modified problem, but because the segment MP is now a chord of C_1, the proof becomes somewhat easier.

4408. *Proposed by Leonard Giugiuc, Dan Stefan Marinescu and Daniel Sitaru.*

Let $\alpha \in (0, 1] \cup [2, \infty)$ be a real number and let a, b and c be non-negative real numbers with $a + b + c = 1$. Prove that

$$a^\alpha + b^\alpha + c^\alpha + 1 \geq (a + b)^\alpha + (b + c)^\alpha + (c + a)^\alpha.$$

We received 4 submissions. One of the submitted solutions was incomplete. We present the proof by Ioannis D. Sfikas.

We give a proof based on the following proposition by Leonard Giugiuc. (See *Hlawka's Inequalities for a class of functions*, Romanian Mathematical Magazine, 2016, by Daniel Sitaru and Leonard Giugiuc.)

Proposition: Let $f(x) : [0, \infty) \to \infty$ be a differentiable function such that $f(0) = 0$ and $f'(x)$ is convex. Then for all nonnegative $x, y, z \in \mathbb{R}$,

$$f(x) + f(y) + f(z) + f(x + y + z) \geq f(x + y) + f(y + z) + f(z + x).$$

Proof of the current problem: Define $f(x) : [0, \infty) \to \mathbb{R}$ by $f(x) = x^\alpha$. Then $f(0) = 0$ and $f'''(x) = \alpha(\alpha - 1)(\alpha - 2)x^{\alpha-3} \geq 0$ since $(\alpha - 1)(\alpha - 2) \geq 0$ for $\alpha \in (0, 1] \cup [2, \infty)$.

Hence $f'(x)$ is convex. We then have, by the Proposition, that

$$f(a) + f(b) + f(c) + f(a + b + c) \geq f(a + b) + f(b + c) + f(c + a)$$

so

$$a^\alpha + b^\alpha + c^\alpha + 1 \geq (a + b)^\alpha + (b + c)^\alpha + (c + a)^\alpha$$

follows.

Editor's comments: It is easy to find a counterexample to show that the proposed inequality needs not be true if $\alpha \in (1, 2)$; e.g., if $\alpha = \frac{3}{2}, a = b = \frac{2}{5}, c = \frac{1}{5}$, then

$$c^\alpha + b^\alpha + c^\alpha + 1 \approx 2(0.4)^{1.5} + (0.2)^{1.5} + 1 = 1.5954\ldots$$

while

$$(a + b)^\alpha + (b + c)^\alpha + (c + a)^\alpha \approx (0.8)^{1.5} + 2(0.6)^{1.5} = 1.6451\ldots,$$

so LHS < RHS.

4409. *Proposed by Christian Chiser.*

Let A and B be two matrices in $M_2(\mathbb{R})$ such that $A^2 = O_2$ and B is invertible. Prove that the polynomial $P = \det(xB^2 - AB + BA)$ has all integer roots.

We received 6 submissions of which 4 were correct and complete. As stated this problem is false and 4 counterexamples were provided. The solution by Ivko Dimitrić featured here provides a general solution for the roots and when such roots are integer.

The statement is shown to be false by the following counter-example with

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

For these matrices, $A^2 = O_2$,

$$xB^2 - AB + BA = \begin{bmatrix} x & 1 \\ 1 & 4x \end{bmatrix}$$

and the roots of $P(x) = 4x^2 - 1$ are non-integers, $x = \pm\frac{1}{2}$.

Nevertheless, it is possible to determine the roots of P in general and examine when they will be integers.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. From

$$A^2 = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

we see that if $a + d \neq 0$ then $b = c = 0$, which immediately yields also $a = d = 0$ and $a+d = 0$, a contradiction! Thus, $\operatorname{tr} A = a+d = 0$, so $d = -a$ and $a^2+bc = 0$.

Let $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with $a^2 = -bc$. Then we compute

$$\begin{aligned} -AB + BA &= \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a & b \\ c & -a \end{bmatrix} - \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \\ &= \begin{bmatrix} cq - br & b(p-s) - 2aq \\ 2ar - c(p-s) & br - cq \end{bmatrix}. \end{aligned}$$

Further,

$$xB^2 = x \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} (p^2+qr)x & q(p+s)x \\ r(p+s)x & (s^2+qr)x \end{bmatrix}$$

Hence,

$$xB^2 - AB + BA = \begin{bmatrix} (p^2+qr)x - (br - cq) & q(p+s)x + b(p-s) - 2aq \\ r(p+s)x - c(p-s) + 2ar & (s^2+qr)x + (br - cq) \end{bmatrix}$$

and

$$\begin{aligned}P(x) = {} & [(p^2+qr)x-(br-cq)]\,[(s^2+qr)x+(br-cq)] \\ & -[q(p+s)x+b(p-s)-2aq]\,[r(p+s)x-c(p-s)+2ar].\end{aligned}$$

The coefficient of x^2 in this quadratic trinomial is

$$(p^2+qr)(s^2+qr)-qr(p+s)^2=(ps-qr)^2=(\det B)^2$$

and the coefficient of x is reduced to

$$\begin{aligned}&(p^2+qr)(br-cq)-(s^2+qr)(br-cq)\\ &\qquad +q(p+s)[c(p-s)-2ar]+r(p+s)[2aq-b(p-s)]\\ &=(br-cq)(p^2-s^2)+(p+s)[cq(p-s)-br(p-s)]\\ &=(br-cq)(p^2-s^2)-(p+s)(p-s)(br-cq)\\ &=0.\end{aligned}$$

Finally, the constant term of $P(x)$ equals

$$\begin{aligned}&-(br-cq)^2+[b(p-s)-2aq]\,[c(p-s)-2ar]\\ &=-(br-cq)^2+bc(p-s)^2-2a(br+cq)(p-s)+4a^2qr\\ &=-(br-cq)^2+(br+cq)^2-[a(p-s)+(br+cq)]^2+4a^2qr\\ &=-[a(p-s)+(br+cq)]^2+4bcqr+4a^2qr\\ &=-[a(p-s)+(br+cq)]^2,\end{aligned}$$

since $bc=-a^2$. Hence,

$$P(x)=(ps-qr)^2x^2-[a(p-s)+(br+cq)]^2$$

has roots

$$x=\pm\frac{a(p-s)+(br+cq)}{ps-qr}.$$

The roots are integer if and only if the quotient on the right hand side is an integer, in particular, when $A,B\in M_2(\mathbb{Z})$ and $A^2=O_2,\ \det B=\pm1$, but in general the roots are non-integers.

4410. *Proposed by Daniel Sitaru.*

Prove that

$$\int_0^{\frac{\pi}{4}}\sqrt{\sin 2x}dx<\sqrt{2}-\frac{\pi}{4}.$$

We received 6 correct solutions. There were 10 additional solutions that can be considered weakly correct in that either they obtained a different upper bound for the integral and then gave a numerical argument that this did not exceed the desired *bound, or based estimates on a series expansion. There was one incorrect solution. We present two solutions following different approaches.*

Solution 1, by Michel Bataille and Àngel Plaza (independently).

The substitution $u = (\pi/4) - x$ leads to

$$\int_0^{\pi/4} \sqrt{\sin 2x}\, dx = \int_0^{\pi/4} \sqrt{\cos 2u}\, du.$$

From the Cauchy-Schwarz Inequality,

$$1 + \sqrt{\cos 2x} < \sqrt{2}(1 + \cos 2x)^{1/2} = 2\cos x.$$

Therefore

$$\frac{\pi}{4} + \int_0^{\pi/4} \sqrt{\sin 2x}\, dx = \int_0^{\pi/4} (1 + \sqrt{\cos 2x})\, dx < 2\int_0^{\pi/4} \cos x dx = \sqrt{2}.$$

The result follows.

Solution 2, Brian Bradie and Daniel Vicaru (independently).

By the Root-Mean-Square (or the Jensen) Inequality,

$$\frac{1 + \sqrt{\sin 2x}}{2} < \sqrt{\frac{1 + \sin 2x}{2}} = \frac{\cos x + \sin x}{\sqrt{2}} = \sin\left(x + \frac{\pi}{4}\right).$$

Hence

$$\frac{\pi}{4} + \int_0^{\pi/4} \sqrt{\sin 2x}\, dx = \int_0^{\pi/4} (1 + \sqrt{\sin 2x})\, dx = 2\left[-\cos\left(x + \frac{\pi}{4}\right)\right]_0^{\pi/4} = \sqrt{2},$$

from which the result follows.

4411. *Proposed by Michel Bataille.*

Let n be a positive integer. Find the largest constant C_n such that

$$\frac{(xy)^n}{z^{n+1}} + \frac{(yz)^n}{x^{n+1}} + \frac{(zx)^n}{y^{n+1}} \geq C_n \left(\max(x, y, z)\right)^{n-1}$$

holds for all real numbers x, y, z satisfying $xyz > 0$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$.

We received 4 solutions, 3 of which were correct. We present the solution by Walther Janous.

The two conditions $xyz > 0$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$ imply that two of the three variables x, y, and z have to be negative. As the inequality is symmetric, we let $x > 0$.

Then, say, $y = -s$, and $z = -t$, with s and t positive. But then $\max\{x, y, z\} = x$, and $\frac{1}{x} - \frac{1}{s} - \frac{1}{t} = 0$ leads to

$$x = \frac{st}{s+t}.$$

The inequality under consideration is successively equivalent to

$$\frac{(-sx)^n}{(-t)^{n+1}} + \frac{(st)^n}{x^{n+1}} + \frac{(-tx)^n}{(-s)^{n+1}} \geq C_n \cdot x^{n-1}$$

$$\frac{(st)^n}{x^{n+1}} - \left(\frac{s^n}{t^{n+1}} + \frac{t^n}{s^{n+1}}\right) \cdot x^n \geq C_n \cdot x^{n-1}$$

$$\frac{(s+t)^{n+1}}{st} - \frac{s^{2n+1} + t^{2n+1}}{(st) \cdot (s+t)^n} \geq C_n \cdot \left(\frac{st}{s+t}\right)^{n-1}$$

$$\frac{(s+t)^{2n}}{(st)^n} - \frac{s^{2n+1} + t^{2n+1}}{(st)^n \cdot (s+t)} \geq C_n.$$

As the left-hand expression is homogeneous of degree 0, we may and do let $t = 1$, resulting in the inequality

$$\frac{(t+1)^{2n}}{t^n} - \frac{t^{2n+1}+1}{t^n \cdot (t+1)} \geq C_n$$

for $t > 0$; that is, upon expanding, we have successively

$$\frac{\sum_{j=0}^{2n} \binom{2n}{j} \cdot t^j - \sum_{j=0}^{2n} t^j}{t^n} \geq C_n,$$

$$\frac{\sum_{j=1}^{2n-1} \left(\binom{2n}{j} - 1\right) \cdot t^j}{t^n} \geq C_n,$$

$$\sum_{j=1}^{n-1} \left(\binom{2n}{j} - 1\right) \cdot \left(t^{n-j} + \frac{1}{t^{n-j}}\right) + \left(\binom{2n}{n} - 1\right) \geq C_n.$$

Since $w + 1/w \geq 2$ for all $w > 0$, the left-hand sum attains its least value for $t = 1$. Therefore, the best constant C_n has the value

$$C_n = \frac{(1+1)^{2n}}{1^n} - \frac{1^{2n+1}+1}{1^n \cdot (1+1)} = 2^{2n} - 1,$$

and the proof is complete.

4412. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with incenter I. If I_a, I_b, I_c are the excenters of ABC, show that $\overrightarrow{II_a} + \overrightarrow{II_b} + \overrightarrow{II_c} = \overrightarrow{0}$ if and only if ABC is equilateral.

We received 10 submissions, all of which were correct, and we present the solution by Cristóbal Sánchez-Rubio with some details added by the editor.

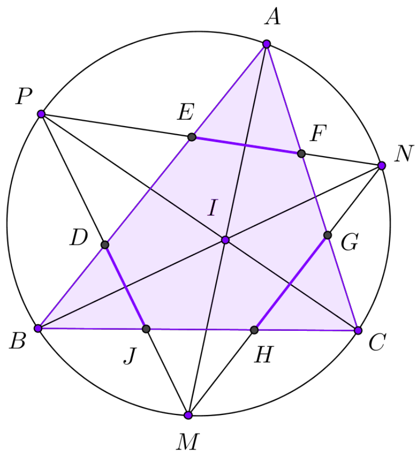
The desired result follows quickly from three familiar theorems; each is elementary and easy to prove for an arbitrary triangle ABC (acute or not).

1. For a point P in the plane of triangle $I_aI_bI_c$, $\overrightarrow{PI_a} + \overrightarrow{PI_b} + \overrightarrow{PI_c} = \overrightarrow{0}$ if and only if P is the centroid of $\Delta I_aI_bI_c$.
2. The incenter of the given triangle ABC is the orthocenter of $\Delta I_aI_bI_c$.
3. A triangle is equilateral if and only if its centroid and orthocenter coincide.

Consequently, $\overrightarrow{II_a} + \overrightarrow{II_b} + \overrightarrow{II_c} = \overrightarrow{0}$ if and only if $\Delta I_aI_bI_c$ is equilateral. But if either triangle $I_aI_bI_c$ or ABC is equilateral, its sides would be parallel to the sides of the other, whence the other would also be equilateral.

4413. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let ABC be a triangle with incenter I and circumcircle ω. The lines AI, BI, CI intersect ω a second time at M, N, P, respectively. Also suppose that NP intersects AB and AC at E and F, respectively. We define points G, H, J and D analogously (see the picture). Show that if $EF = GH = JD$, then triangle ABC is equilateral.



We received 9 submissions, all correct, and present a composite of similar solutions from Prithwijit De and Jirapat Kaewkam, done independently.

Let $T = AI \cap NP$. Observe that in triangle AIP,

$$\angle APT = B/2 = \angle IPT \quad \text{and} \quad \angle IAP = A/2 + C/2 = \angle AIP.$$

Thus, ΔAIP is isosceles and PT bisects its vertex angle, so that $PT \perp AI$ and $AT = TI = AI/2$. Moreover, in ΔAEF the bisector AT of the angle at A is perpendicular to the base, whence $ET = TF = EF/2$. Thus

$$EF = 2ET = 2AT\tan(A/2) = \frac{AI\sin(A/2)}{\cos(A/2)} = \frac{r}{\cos(A/2)},$$

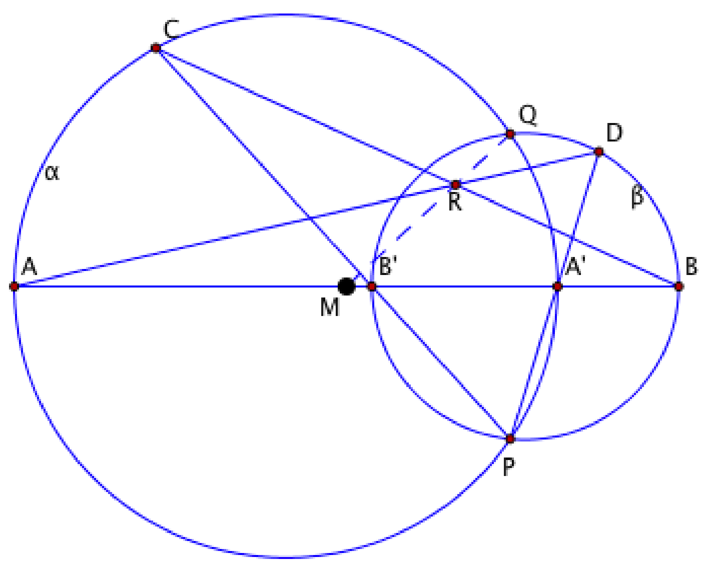
where r is the inradius of triangle ABC. Similarly,

$$GH = \frac{r}{\cos(C/2)} \quad \text{and} \quad JD = \frac{r}{\cos(B/2)}.$$

It follows that $EF = GH = JD$ if and only if $\cos(A/2) = \cos(B/2) = \cos(C/2)$, which (because $0 < \frac{A}{2}, \frac{B}{2}, \frac{C}{2} < 90°$) is equivalent to $A = B = C = 60°$. In other words, ΔABC is equilateral if and only if $EF = GH = JD$.

4414. *Proposed by Konstantin Knop.*

Let α and β be a pair of circles that intersect in points P and Q, and let the diameter AA' of α lie on the same line as the diameter BB' of β such that the end points lie in the order $AB'A'B$. Suppose that PB' intersects α again at the point C, that PA' intersects β again at D, and that the lines AD and BC intersect at R. Prove that the line QR intersects the segment AB at its midpoint.



We received 2 solutions to this problem. However, both submissions utilized brute force calculations to achieve the result. We leave the problem open in hopes to receive a more insightful solution. Please email your submissions directly to `crux-editors@cms.math.ca` .

4415. *Proposed by Titu Zvonaru.*

Let ABC be an acute-angled triangle with $AB < AC$, where AD is the altitude from A, O is the circumcenter and M and N are the midpoints of the sides BC and AB, respectively. The line AO intersects the line MN at X. Prove that DX is parallel to OC.

We received 13 correct solutions. Of those, 7 gave a synthetic argument; 4 used analytic geometry and 1 used barycentric coordinates.

Solution 1, by Dimitrić Ivko.

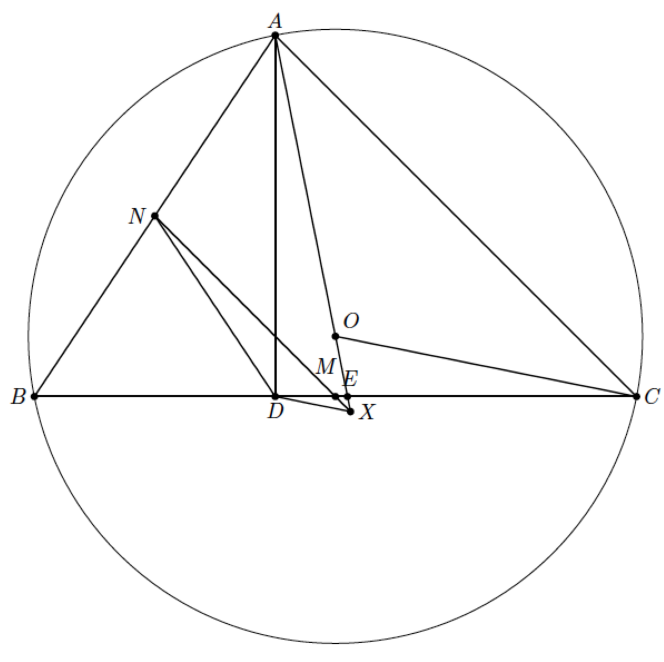
Let $\alpha = \angle CAB$, $\beta = \angle ABC$, $\gamma = \angle BCA$. Since $\angle AOC = 2\beta$ and $OA = OC$, then $\angle XAC = 90^\circ - \beta$. Since $NM \| AC$, then $\angle NXA = \angle XAC = 90^\circ - \beta$. Since $BN = ND$, then $\angle NDA = 90^\circ - \angle NDB = 90^\circ - \beta$.

Thus, $\angle NXA = \angle NDA$, and so $ANDX$ is cyclic. Hence

$$\angle AXD = 180^\circ - \angle AND = \angle BND = 180^\circ - 2\beta = \angle XOC.$$

It follows that $DX \| OC$.

(*Note from Sushanth Sathish Kumar:* Once $ANDX$ is proved to be cyclic, $DX \| OC$ follows from $\angle NXD = \angle NAD = 90^\circ - \beta = \angle OCA$ and $NX \| AC$.)



Solution 2, by Vijaya Prasad Nalluri.

Let BC and AX intersect at E. Since $NX \| AC$, triangles MEX and CEA are similar, so that $XE : AE = ME : EC$. Since $AD \| OM$, triangles MOE and

DAE are similar, so that $AE : OE = DE : ME$. Hence $XE : OE = DE : EC$. Therefore, triangles XED and OEC are similar, so that $\angle DXE = \angle COE$. This equality of alternate angles implies that $DX \| OC$.

Solution 3, by Prithwijit De.

Assign coordinates: $O \sim (0,0)$, $A \sim (p,q)$, $B \sim (-b,k)$, $C \sim (b,k)$. Then $M \sim (0,k)$ and $D \sim (p,k)$. The equation of the line AO is $y = (q/p)x$. Since MN passes through $(0,k)$ and has the same slope as AC, its equation is

$$y = k + \left(\frac{k-q}{b-p}\right)x.$$

The point X where MN and AO intersect has coordinates

$$\left(\frac{kp(b-p)}{qb-pk}, \frac{kq(b-p)}{qb-pk}\right).$$

The slope of the line DX is

$$\frac{\frac{kq(b-p)}{qb-pk} - k}{\frac{kp(b-p)}{qb-pk} - p} = \frac{kp(k-q)}{pb(k-q)} = \frac{k}{b},$$

which is the slope of AC. The result follows.

4416. *Proposed by Nguyen Viet Hung.*

Let ABC be an acute triangle with orthocentre H. Denote by r_a, r_b, r_c the exradii opposite the vertices A, B, C, and by r_1, r_2, r_3 the inradii of triangles BHC, CHA, AHB, respectively. Prove that

$$r_1 + r_2 + r_3 + r_a + r_b + r_c = a + b + c.$$

We received 9 submissions, all correct, and present the solution by Kee-Wai Lau.

We start with standard formulas for the inradius and an exradius of a triangle ABC in terms of its circumradius R:

$$r = R(\cos A + \cos B + \cos C - 1) \tag{1}$$

and

$$r_a = R(1 + \cos B + \cos C - \cos A). \tag{2}$$

By (2) and the corresponding expressions for r_b and r_c we obtain

$$r_a + r_b + r_c = R(3 + \cos A + \cos B + \cos C). \tag{3}$$

We now show that

$$r_1 = R(\sin B + \sin C - \cos A - 1). \tag{4}$$

Note that the angles of ΔBCH are $\frac{\pi}{2}-C$, $\frac{\pi}{2}-B$, and $B+C$, while its circumradius is
$$\frac{BC}{2\sin(B+C)}=\frac{a}{2\sin A}=R.$$
We get formula (4) by replacing A by $B+C$, B by $\frac{\pi}{2}-C$, and C by $\frac{\pi}{2}-B$ in (1). This, together with the corresponding expressions for r_2 and r_3, gives us
$$r_1+r_2+r_3=R(2\sin A+2\sin B+2\sin C-\cos A-\cos B-\cos C-3). \qquad (5)$$
Adding together (3) and (5), we obtain
$$r_1+r_2+r_3+r_a+r_b+r_c=2R(\sin A+\sin B+\sin C)=a+b+c,$$
as desired.

4417. *Proposed by Dan Stefan Marinescu, Daniel Sitaru and Leonard Giugiuc.*

Let a, b and c be positive real numbers such that $abc \geq 1$. Further, let x, y and z be real numbers such that $xy+yz+zz \geq 3$. Prove that
$$(y^2+z^2)a+(z^2+x^2)b+(x^2+y^2)c \geq 6.$$

We received 4 solutions, 3 of which were correct. We present the solution by Walther Janous.

We shall prove a more general result, namely:
$$\left(y^2+z^2\right)a+\left(z^2+x^2\right)b+\left(x^2+y^2\right)c \geq 2(xy+yz+zx).$$

This will follow if we show that Q, defined by
$$Q=(b+c)x^2+(c+a)y^2+(a+b)z^2-2xy-2yz-2zx$$
is a positive semi-definite quadratic form. But its corresponding symmetric matrix equals
$$M=\begin{bmatrix} b+c & -1 & -1 \\ -1 & c+a & -1 \\ -1 & -1 & a+b \end{bmatrix}.$$
We thus have to check only its three principal minors.

- The inequality $b+c>0$ is clear.
- Next we have to show that
$$\begin{vmatrix} b+c & -1 \\ -1 & c+a \end{vmatrix} > 0,$$
that is
$$ab+ac+bc+c^2>1.$$
By the AM-GM inequality, we get even more:
$$ab+ac+bc+c^2>ab+ac+bc \geq 3(abacbc)^{1/3}=3(abc)^{2/3} \geq 3.$$

- Finally, the inequality $\det M \geq 0$ has to be verified; that is,

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 2abc - 2a - 2b - 2c - 2 \geq 0,$$

or equivalently,

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 2abc \geq 2a + 2b + 2c + 2.$$

Because of $abc \geq 1$, it certainly will follow from

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 \geq 2a + 2b + 2c. \tag{1}$$

Since

$$\frac{5}{6}\left(a^2b + a^2c\right) + \frac{1}{6}\left(b^2c + bc^2\right) = (b+c) \cdot \left(\frac{5}{6}a^2 + \frac{1}{6}bc\right),$$

the left-hand expression in (1) can be written as

$$(b+c) \cdot \left(\frac{5}{6}a^2 + \frac{1}{6}bc\right) + (c+a) \cdot \left(\frac{5}{6}b^2 + \frac{1}{6}ca\right) + (a+b) \cdot \left(\frac{5}{6}c^2 + \frac{1}{6}ab\right).$$

But the AM-GM inequality yields

$$\begin{aligned}(b+c) \cdot \left(\frac{5}{6}a^2 + \frac{1}{6}bc\right) &\geq 2\sqrt{bc} \cdot \left(a^2\right)^{5/6} \cdot (bc)^{1/6} \\ &= 2a^{5/3}(bc)^{2/3} \\ &= 2a(abc)^{2/3} \geq 2a,\end{aligned}$$

and two similar inequalities for the other two summands. This completes the proof.

Remark: It is a bit disturbing to have a, b, c limited by the constraint $abc \geq 1$. We shall remove it as follows. Let a, b, and c be arbitrary positive real numbers. Then the three numbers

$$a_1 = \frac{a}{(abc)^{1/3}}, b_1 = \frac{b}{(abc)^{1/3}}, c_1 = \frac{c}{(abc)^{1/3}}$$

satisfy the condition

$$a_1b_1c_1 = 1 \geq 1,$$

whence by what we have already shown,

$$(b_1 + c_1)\, x^2 + (c_1 + a_1)\, y^2 + (a_1 + b_1)\, z^2 \geq 2(xy + yz + zx);$$

that is, there holds even more generally

$$(b+c)x^2 + (c+a)y^2 + (a+b)z^2 \geq 2(abc)^{1/3}(xy + yz + zx)$$

for all positive real numbers a, b, and c and all real numbers x, y, and z. Furthermore, the various applications of the AM-GM inequality show that equality occurs if and only if $a = b = c$ and $x = y = z$.

4418. *Proposed by Daniel Sitaru.*

Consider a convex cyclic quadrilateral with sides a, b, c, d and area S. Prove that

$$\frac{(a+b)^5}{c+d} + \frac{(b+c)^5}{d+a} + \frac{(c+d)^5}{a+b} + \frac{(d+a)^5}{b+c} \geq 64S^2.$$

We received 7 correct solutions. We present 5 of them.

We make some preliminary remarks. The formula for the area S of a quadrilateral with sides a, b, c, d and perimeter $2s = a+b+c+d$ is

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\theta},$$

where θ is half the sum of two opposite angles. This is dominated by the area of a cyclic quadrilateral with the same sides, namely

$$\begin{aligned}
&\sqrt{(s-a)(s-b)(s-c)(s-d)}\\
&= \frac{1}{4}\sqrt{(b+c+d-a)(c+d+a-b)(d+a+b-c)(a+b+c-d)}\\
&= \frac{1}{4}\sqrt{[(a+b)^2-(c-d)^2][(c+d)^2-(a-b)^2]}\\
&= \frac{1}{4}\sqrt{[(a+c)^2-(b-d)^2][(b+d)^2-(a-c)^2]}.
\end{aligned}$$

The statement of the problem remains true for noncyclic quadrilaterals.

Solution 1, by Oliver Geupel.

Let

$$(w, x, y, z) = (s-a, s-b, s-c, s-d).$$

Then

$$(a+b, b+c, c+d, d+a) = (y+z, z+w, w+x, x+y).$$

Applying the arithmetic-geometric means inequality twice, we find that

$$\begin{aligned}
&\frac{(a+b)^5}{c+d} + \frac{(b+c)^5}{d+a} + \frac{(c+d)^5}{a+b} + \frac{(d+a)^5}{b+c}\\
&\qquad = \frac{(y+z)^5}{w+x} + \frac{(z+w)^5}{x+y} + \frac{(w+x)^5}{y+z} + \frac{(x+y)^5}{z+w}\\
&\qquad \geq 4(y+z)(z+w)(w+x)(x+y)\\
&\qquad \geq 4(2\sqrt{yz})(2\sqrt{zw})(2\sqrt{wx})(2\sqrt{xy})\\
&\qquad = 64xyzw \geq 64S^2.
\end{aligned}$$

Equality holds if and only if the quadrilateral is a square.

Solution 2, by Šefket Arslanagić.

By the arithmetic-geometric means inequality,

$$\begin{aligned} S &\leq \sqrt{(s-a)(s-b)}\sqrt{(s-c)(s-d)} \\ &\leq \frac{1}{4}(2s-a-b)(2s-c-d) = \frac{1}{4}(c+d)(a+b). \end{aligned}$$

Similarly, $S \leq \frac{1}{4}(b+c)(a+d)$. Therefore

$$\begin{aligned} 64S^2 &= 4(16S^2) \\ &\leq 4(a+b)(b+c)(c+a)(d+a) \\ &= 4\left[\frac{(a+b)^5}{c+d}\cdot\frac{(b+c)^5}{d+a}\cdot\frac{(c+d)^5}{a+b}\cdot\frac{(d+a)^5}{b+c}\right]^{1/4} \\ &\leq \frac{(a+b)^5}{c+d} + \frac{(b+c)^5}{d+a} + \frac{(c+d)^5}{a+b} + \frac{(d+a)^5}{b+c}. \end{aligned}$$

Solution 3, by C.R. Pranesachar.

By the arithmetic-geometric means inequality,

$$\begin{aligned} \frac{(a+b)^5}{c+d} + \frac{(c+d)^2}{a+b} &\geq 2[(a+b)^2(c+d)^2] \\ &\geq 2[(a+b)^2-(c-d)^2][(c+d)^2-(a-b)^2] \\ &\geq 32S^2. \end{aligned}$$

A similar inequality holds for the other two terms of the left side and the result follows.

Solution 4, by Leonard Giugiuc and Digby Smith, independently. We have:

$$\begin{aligned} 64S^2 &= 64(s-a)(s-b)(s-c)(s-d) \\ &\leq 64\left[\frac{(s-a)+(s-b)+(s-c)+(s-d)}{4}\right]^4 \\ &= 64\left(\frac{2s}{4}\right)^4 \\ &= 4s^4. \end{aligned}$$

From an instance of the Hölder inequality, for positive x, y, z, t, m, n, p, q,

$$\left(\frac{x^5}{m}+\frac{y^5}{n}+\frac{z^5}{p}+\frac{t^5}{q}\right)(m+n+p+q)(1+1+1+1)^3 \geq (x+y+z+t)^5,$$

applied to

$$(x, y, z, t; m, n, p, q) = (a + b, b + c, c + d, d + a; c + d, d + a, a + b, b + c),$$

we find that the left side is not less than

$$\frac{2^5(a+b+c+d)^5}{4^3 \cdot 2(a+b+c+d)} = \frac{2^{10}s^5}{2^8 s} = 4s^4 \geq 64S^2.$$

Solution 5, by Walther Janous.

We prove a more general result: *Let $p > q > 0$ and $p + q \geq 1$. Then*

$$\frac{(a+b)^p}{(c+d)^q} + \frac{(b+c)^p}{(d+a)^q} + \frac{(c+d)^p}{(a+b)^q} + \frac{(d+a)^p}{(b+c)^q} \geq 2^{p-q+2}S^{(p-q)/2}.$$

Applying the arithmetic-geometric means inequality to the denominator yields

$$\begin{aligned}\frac{(a+b)^p}{(c+d)^q} + \frac{(c+d)^p}{(a+b)^q} &= \frac{(a+b)^{p+q} + (c+d)^{p+q}}{[(a+b)(c+d)]^q} \\ &\geq 2^{2q} \cdot \frac{[(a+b)^{p+q} + (c+d)^{p+q}]}{(a+b+c+d)^{2q}},\end{aligned}$$

with an analogous inequality for the other two terms on the left side. Using the convexity of x^{p+q}, we see that the left side is not less than

$$\begin{aligned}&2^{2q}\left[\frac{(a+b)^{p+q} + (b+c)^{p+q} + (c+d)^{p+q} + (d+a)^{p+q}}{(a+b+c+d)^{2q}}\right] \\ &\geq \frac{2^{2q} \cdot 4}{(a+b+c+d)^{2q}}\left[\frac{(a+b)+(b+c)+(c+d)+(d+a)}{4}\right]^{p+q} \\ &= \frac{2^{2q+2}}{(a+b+c+d)^{2q}}\left[\frac{a+b+c+d}{2}\right]^{p+q} \\ &= 2^{q-p+2}(a+b+c+d)^{p-q}.\end{aligned}$$

On the other hand, from the AM-GM inequality [as in Solution 4],

$$S \leq \frac{(a+b+c+d)^2}{4}$$

whereupon

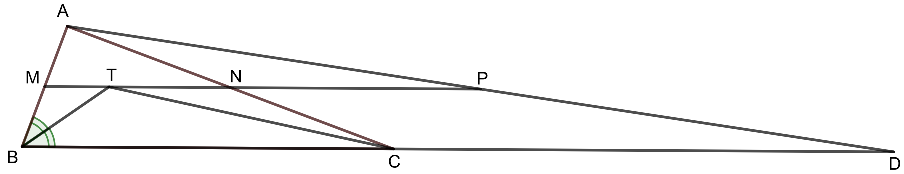
$$\begin{aligned}2^{p-q+2}S^{(p-q)/2} &\leq 2^{p-q+2}\left[\frac{(a+b+c+d)^{p-q}}{2^{2(p-q)}}\right] \\ &= 2^{q-p+2}(a+b+c+d)^{p-q}.\end{aligned}$$

The result follows.

4419. *Proposed by Michel Bataille.*

Let ABC be a triangle with $\angle BAC = 90°$. Let D on the hypotenuse BC produced beyond C be such that $CD = CB+BA$. The internal bisector of $\angle ABC$ intersects the line through the midpoints of AB and AC at T. Prove that $\angle TCA = \angle CDA$.

We received 13 solutions. We present the solution by Mihai Miculiţa and Titu Zvonaru.



Use a, b and c to denote the lengths of the sides of the triangle. Let M and N be the midpoints of AB and AC, respectively, and P be the intersection of line MN with line AD. Note that $MN||BC$ and $MN = \frac{a}{2}$; also, P must be the midpoint of AD.

Since BT is the bisector of $\angle CBA$, so $\angle CBT = \angle TBA$, and $MN||BC$ gives us $\angle MTB = \angle TBC$. So $\triangle MTB$ is isosceles, giving us $MT = MB = c/2$, and hence $NT = MN - MT = \frac{a-c}{2}$.

Since NP joins the midpoints of AC and AD, using $CD = a+c$ we get $NP = \frac{a+c}{2}$.

Thus

$$NT \cdot NP = NA \cdot NC \quad \iff \quad (a-c)(a+c) = b^2,$$

which holds by the Pythagorean Theorem. We deduce that the quadrilateral $PCTA$ is cyclic, thus $\angle TCA = \angle TPA$.

Finally, since $NP||CD$ we have $\angle TPA = \angle CDA$, allowing us to conclude $\angle TCA = \angle CDA$.

4420. *Proposed by Leonard Giugiuc and Marian Dinca.*

Let $A_0A_1 \ldots A_{n-1}$, $n \geq 10$ be a regular polygon inscribed in a circle of radius r centered at O. Consider the closed disks $\omega(A_k), k = 0, \ldots, n-1$ centered at A_k of radius r. Prove that

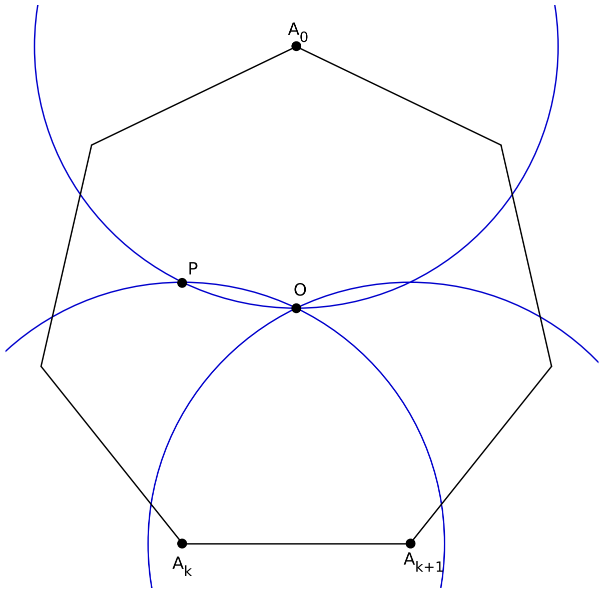
$$\bigcap_{k=0}^{n-1} \omega(A_k) = \{O\}.$$

We received 4 solutions. Presented is the one by Walther Janous, lightly edited.

We show that the statement holds for all $n \geq 3$. First of all, it's clear that O is an element of the intersection under consideration. We now distinguish two cases.

If n is even, then the disks $\omega(A_0)$ and $\omega(A_{n/2})$ are tangent to each other.

If n is odd, let $n = 2k + 1$. We consider $\omega(A_0)$ and its opposite disks $\omega(A_k)$ and $\omega(A_{k+1})$. Suppose w.l.o.g. that A_0 is north of O and the vertices of the polygon are ordered counterclockwise. Then the boundary circles of $\omega(A_0)$ and $\omega(A_k)$ intersect in O (the southernmost point of $\omega(A_0)$ and a point P northwest of O.



Thus the intersection $\omega(A_0) \cap \omega(A_k)$ is contained in the western hemisphere. By a similar argument the intersection $\omega(A_0) \cap \omega(A_{k+1})$ is contained in the eastern hemisphere. Therefore the intersection $\omega(A_0) \cap \omega(A_k) \cap \omega(A_{k+1})$ consists of only O and the claim follows.