OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(1), p. 17–18; 45(2): 69–71.

OC411. Show that for all integers $k > 1$ there is a positive integer m less than k^2 such that $2^m - m$ is divisible by k.

Originally 2017 Hungary Math Olympiad, 3rd Problem, 3rd Category, Final Round.

We received no submissions for this problem.

OC412. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y

$$f(y - xy) = f(x)y + (x - 1)^2 f(y).$$

Originally 2017 Czech-Slovakia Math Olympiad, 3rd Problem, Final Round.

We received 5 submissions of which 4 were correct. We present the solution by Sundara Narasimhan.

We evaluate the relation at $x = 0$ and $y = 1$ to find $0 = f(0)$.

We evaluate the relation at $x = 1$ and $y = 1$ to find $f(0) = f(1)$.

We evaluate the relation at $x = x$ and $y = 1$, and use $f(0) = f(1) = 0$ to find $f(1 - x) = f(x)$.

We make the substitution $1-x = t$ in the original relation, and use $f(1-x) = f(x)$ to get for any $t \in \mathbb{R}$ and $y \in \mathbb{R}$

$$f(yt) = f(t)y + t^2 f(y).$$

We interchange y and t to get $f(ty) = f(y)t + y^2 f(t)$. Since $f(ty) = f(yt)$, we find that for any $t \in \mathbb{R}$ and $y \in \mathbb{R}$

$$(t^2 - t)f(y) = (y^2 - y)f(t).$$

We take $t = 2$ in the last relation to find that for any $y \in \mathbb{R}$

$$f(y) = \frac{f(2)}{2}(y^2 - y).$$

Therefore, the solutions of our functional equation must be of the form

$$f(x) = c(x^2 - x),$$

for some real constant c. In fact, we can check that any function of this form is a solution of the original relation. We established that the set of all functions that satisfy the original relation are $f(x) = c(x^2 - x)$, with c being a real constant.

OC413. To each sequence consisting of n zeros and n ones is assigned a number which is the number of largest segments with the same digits in it (for example, the sequence 00111001 has 4 such segments 00, 111, 00, 1). For each n, add all the numbers assigned to each sequence. Prove that the resulting sum is equal to

$$(n+1)\binom{2n}{n}.$$

Originally 2017 Czech-Slovakia Math Olympiad, 4th Problem, Final Round.

We received one submission. We present the solution of Kathleen Lewis.

The total number of distinct sequences of n zeroes and n ones is $\binom{2n}{n}$. The number of largest same-digit segments of such sequence has a range between 2 and $2n$. The minimum number of 2 is displayed by two sequences that have all zeros together and all ones together

$$(000\ldots111\ldots,\ 111\ldots000\ldots).$$

The maximum number is displayed by two sequences that have alternating zeros and ones

$$(101010\ldots,\ 010101\ldots).$$

For a natural number j between 2 and $2n$, let N_j be the number of sequences that have exactly j largest same-digit segments.

First we show that for any j, $N_j = N_{2n+2-j}$, in other words $N_2, N_3, \ldots, N_{2n+1}$, N_{2n+2} are symmetrical about $n+1$. In fact we can calculate N_j.

Case 1. Assume j is even, i.e. $j = 2k$ for some natural number k.

Since the sequence has j same-digit blocks, k of these are blocks of zeroes and the remaining k are blocks of ones. The sequence is uniquely determined by the points where we cut the original list of n zeros and the original list of n ones. $k-1$ cuts need to be made to obtain k blocks, and these cuts are selected from $n-1$ links between the original n zeros. Therefore the original sequence of n zeros can be cut in k blocks in $\binom{n-1}{k-1}$ ways. Similarly for ones. Hence

$$N_j = 2\binom{n-1}{k-1}\binom{n-1}{k-1}.$$

The number 2 was added to the above expression to account for whether the sequence starts with zero or one.

Since j is even, it follows that $2n+2-j = 2(n+1-k)$ is even, and

$$N_{2n+2-j} = 2\binom{n-1}{n-k}\binom{n-1}{n-k}$$

Using properties of binomial coefficients

$$N_{2n+2-j} = 2\binom{n-1}{(n-1)-(n-k)}\binom{n-1}{(n-1)-(n-k)} = N_j.$$

Case 2. Assume j is odd, i.e. $j = 2k+1$ for some natural number k.

Since the sequence has j same-digit blocks, $k+1$ of these are blocks of zeroes and the remaining k are blocks of ones, or vice versa k blocks of zeros and $k+1$ blocks of ones. Using arguments that we invoked at case 1 we show that

$$N_j = 2\binom{n-1}{k}\binom{n-1}{k-1}.$$

Since j is odd, it follows that $2n+2-j = 2(n-k)+1$ is odd, and

$$N_{2n+2-j} = 2\binom{n-1}{n-k-1}\binom{n-1}{n-k}$$

Using properties of binomial coefficients

$$N_{2n+2-j} = 2\binom{n-1}{(n-1)-(n-k-1)}\binom{n-1}{(n-1)-(n-k)} = N_j.$$

Now we can proceed to calculate the required sum

$$S = 2N_2 + 3N_3 + \cdots + (2n-1)N_{2n-1} + (2n)N_{2n}.$$

Because $N_j = N_{2n+2-j}$, we have

$$\begin{aligned}2S &= (2N_2 + (2n)N_{2n}) + (3N_3 + (2n-1)N_{2n-1}) + \cdots + ((2n)N_{2n} + 2N_2)\\ &= (2+2n)N_2 + (3+2n-1)N_3 + \cdots + (2n-1+2)N_{2n-1} + (2n+2)N_{2n}\\ &= 2(n+1)(N_2 + N_3 + \cdots + N_{2n-1} + N_{2n}).\end{aligned}$$

However, $N_2 + N_3 + \cdots + N_{2n-1} + N_{2n}$ is the total number of sequences of n zeros and n ones, namely $\binom{2n}{n}$. Therefore, the sum $S = (n+1)\binom{2n}{n}$.

An interesting interpretation of this result is that the average number of largest same-digit segments in a sequence of n zeros and n ones is $n+1$. And this is mainly due to the fact that the distribution of the number of largest same-digit segments is symmetrical about $n+1$.

OC414. Find all prime numbers p and all positive integers a and m such that $a \leq 5p^2$ and $(p-1)! + a = p^m$.

Originally 2017 Bulgaria Math Olympiad, 4th Problem, Grade 9-12, Final Round.

We received only one incomplete submission, which we do not present here.

OC415. Let n be a positive integer and let $f(x)$ be a polynomial of degree n with real coefficients and n distinct positive real roots. Is it possible for some integer $k \geq 2$ and for a real number a that the polynomial

$$x(x+1)(x+2)(x+4)f(x)+a$$

is the k-th power of a polynomial with real coefficients?

Originally 2017 Bulgaria Math Olympiad, 5th Problem, Grade 9-12, Final Round.

We received no submissions for this problem.

OC416. Given an acute nonisosceles triangle ABC with altitudes CD, AE, BF. Points E' and F' are symmetrical to E and F with respect to points A and B, respectively. Take a point C_1 on the ray $\overrightarrow{CD}$ such that $DC_1 = 3CD$. Prove that $\angle E'C_1F' = \angle ACB$.

Originally 2017 Bulgaria Math Olympiad, 6th Problem, Grade 9-12, Final Round.

We received 3 submissions and we present 2 of them.

Solution 1, by Oliver Geupel.

We drop the constraint that triangle ABC is acute and nonisosceles, and prove the result for an arbitrary triangle ABC. Moreover, we prove the stronger result that the triangles ABC and $E'F'C_1$ are similar.

We work in the complex plane. We use lower-case letters to denote the complex-number representations of geometrical points denoted by corresponding upper-case letters. For example a is the complex number assigned to point A. We assume without loss of generality that the points A, B, and C are on the unit circle.

First we recall the result that the foot of the perpendicular from an arbitrary point P to the chord XY of the unit circle is the point specified by the complex number

$$\frac{1}{2}(p+x+y-xy\bar{p}).$$

Hence,

$$d=\frac{1}{2}\left(a+b+c-\frac{ab}{c}\right),\ e=\frac{1}{2}\left(a+b+c-\frac{bc}{a}\right),\ f=\frac{1}{2}\left(a+b+c-\frac{ca}{b}\right).$$

Moreover, since points E' and F' are symmetrical to E and F with respect to points A and B, respectively

$$e'=a+(a-e)=\frac{1}{2}\left(3a-b-c+\frac{bc}{a}\right),\ f'=b+(b-f)=\frac{1}{2}\left(3b-c-a+\frac{ca}{b}\right).$$

Also,

$$c_1=d+3(d-c)=2a+2b-c-\frac{2ab}{c}.$$

Next, we compute $a(c-b)(e'-c_1)$ and $b(c-a)(f'-c_1)$ to find that

$$a(c-b)(e'-c_1) = b(c-a)(f'-c_1)$$
$$= \frac{5}{2}(a^2b+ab^2) + \frac{1}{2}(bc^2-cb^2) + \frac{1}{2}(ac^2-a^2c) - 3abc - 2\frac{a^2b^2}{c}.$$

Thus,

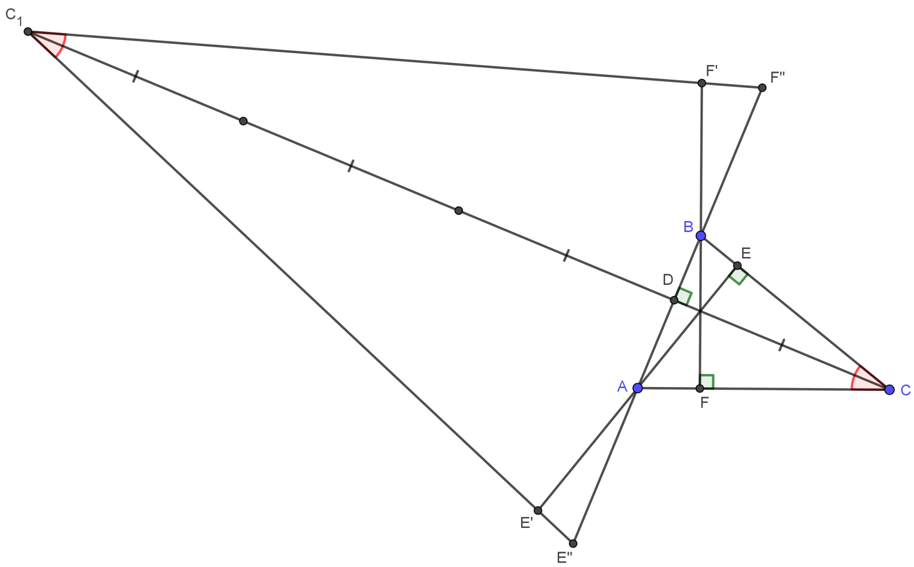
$$\frac{e'-c_1}{f'-c_1} = \frac{b(c-a)}{a(c-b)} = \frac{(1/a)-(1/c)}{(1/b)-(1/c)} = \frac{\bar{a}-\bar{c}}{\bar{b}-\bar{c}}.$$

This equality of complex numbers implies,

$$\frac{C_1E'}{C_1F'} = \frac{CA}{CB} \quad \text{and} \quad \angle E'C_1F' = \angle ACB.$$

This completes the proof.

Solution 2, by Andrea Fanchini.



We use Conway triangle notations: S stands for twice the area of $\triangle ABC$, $S_A = S\cot\angle BAC$, $S_B = S\cot\angle ABC$, and $S_C = S\cot\angle ACB$.

We use barycentric coordinates with reference to the triangle ABC:

$$D(S_B : S_A : 0), \qquad E(0 : S_C : S_B), \qquad F(S_C : 0 : S_A)$$
$$E'(-2a^2 : S_C : S_B), \qquad F'(S_C : -2b^2 : S_A).$$

Since the point C_1 divides the segment CD in the ratio $(-4:3)$, it follows that

$$CC_1/C_1D = (-4)/3 \quad \text{and} \quad C_1(4S_B : 4S_A : -3c^2).$$

Therefore, the lines C_1E', and C_1F' are

$$C_1E' : (S_AS_B + 3S^2)x + 2(S_B^2 + 3S^2)y + 4(a^2S_A + S^2)z = 0,$$

$$C_1F' : 2(S_A^2 + 3S^2)x + (S_AS_B + 3S^2)y + 4(b^2S_B + S^2)z = 0,$$

and the intersection points of these lines with the line AB are

$$E'' = C_1E' \cap AB = \left(2(S_B^2 + 3S^2) : -(S_AS_B + 3S^2) : 0\right),$$

$$F'' = C_1F' \cap AB = \left(S_AS_B + 3S^2 : -2(S_A^2 + 3S^2) : 0\right).$$

We calculate

$$\begin{aligned} \angle E'C_1F' &= \angle F''C_1D + \angle E''C_1D \\ &= \arctan\frac{F''D}{C_1D} + \arctan\frac{E''D}{C_1D} = \arctan\frac{E''F''\cdot C_1D}{C_1D^2 - E''D\cdot F''D}, \end{aligned}$$

where

$$E''F'' = \frac{3\left((a^2S_A + S^2)(2b^2 + S_C) + (b^2S_B + S^2)(2a^2 + S_C)\right)}{c(2a^2 + S_C)(2b^2 + S_C)},$$

$$C_1D = \frac{3S}{c}, \qquad E''D = \frac{3(a^2S_A + S^2)}{c(2a^2 + S_C)}, \qquad F''D = \frac{3(b^2S_B + S^2)}{c(2b^2 + S_C)}$$

Therefore,

$$\begin{aligned} \angle E'C_1F' &= \arctan\frac{S\left((a^2S_A + S^2)(2b^2 + S_C) + (b^2S_B + S^2)(2a^2 + S_C)\right)}{S^2(2a^2 + S_C)(2b^2 + S_C) - (a^2S_A + S^2)(b^2S_B + S^2)} \\ &= \arctan\frac{S_C(8S_CS^2 + 3c^2S^2 + a^2b^2c^2)}{S(8S_CS^2 + 3c^2S^2 + a^2b^2c^2)} \\ &= \arctan\frac{S_C}{S} \\ &= \angle ACB. \end{aligned}$$

OC417. Point M is the midpoint of side BC of a triangle ABC in which $AB = AC$. Point D is the orthogonal projection of M onto side AB. Circle ω is inscribed in triangle ACD and tangent to segments AD and AC at K and L, respectively. Lines tangent to ω which pass through M intersect line KL at X and Y, where points X, K, L and Y lie on KL in this order. Prove that points M, D, X and Y are concyclic.

Originally 2017 Poland Math Olympiad, 5th Problem, Final Round.

We received no submissions for this problem.

OC418. Three sequences $(a_0, a_1, \ldots, a_n)$, $(b_0, b_1, \ldots, b_n)$, $(c_0, c_1, \ldots, c_{2n})$ of nonnegative real numbers are given such that for all $0 \le i, j \le n$ we have $a_i b_j \le (c_{i+j})^2$. Prove that

$$\sum_{i=0}^{n} a_i \cdot \sum_{j=0}^{n} b_j \le \left(\sum_{k=0}^{2n} c_k \right)^2 .$$

Originally 2017 Poland Math Olympiad, 6th Problem, Final Round.

We received no submissions for this problem.

OC419. Prove that there exist infinitely many positive integers m such that there exist m consecutive perfect squares with sum m^3. Determine one solution with $m > 1$.

Originally 2017 Germany Math Olympiad, 6th Problem, Final Round.

We received 6 correct submissions. We present a solution that follows the submissions of the Problem Solving Group of Missouri State University and David Manes. At the end, we include a list of examples by Dominique Mouchet.

We start by computing the difference between m^3 and the sum of m arbitrary consecutive perfect squares:

$$\begin{aligned} m^3 - \sum_{i=a}^{a+m-1} i^2 &= m^3 - \left(\sum_{i=1}^{a+m-1} i^2 - \sum_{i=1}^{a-1} i^2 \right) \\ &= m^3 - \left(\frac{(a+m-1)(a+m)(2a+2m-1)}{6} - \frac{(a-1)a(2a-1)}{6} \right) \\ &= \frac{m\left(4m^2 - 6am + 3m - 6a^2 + 6a - 1\right)}{6}. \end{aligned}$$

Therefore, for any pair of positive integers (a, m) that satisfy

$$f(a, m) = 4m^2 - 6am + 3m - 6a^2 + 6a - 1 = 0,$$

we have m consecutive squares summing to m^3.

Let $a_0 = 1$ and $m_0 = 1$ and recursively define for any $n \ge 1$

$$\begin{aligned} a_n &= 11a_{n-1} + 16m_{n-1} - 5 \\ m_n &= 24a_{n-1} + 35m_{n-1} - 12. \end{aligned} \qquad (1)$$

We prove by induction that $f(a_n, m_n) = 0$, $a_n \ge 1$, $m_n \ge 1$ for all integers $n \ge 1$, and $m_i \ne m_j$ for all integers $i \ne j$.

First it is easy to check $f(a_0, m_0) = f(1, 1) = 0$. Assuming that $f(a_{n-1}, m_{n-1}) = 0$, a routine, but tedious, calculation yields $f(a_n, m_n) = 0$. Second, $a_0, m_0 \ge 1$ and assuming $a_{n-1}, m_{n-1} \ge 1$, it follows that $a_n \ge 11 + 16 - 5 \ge 1$ and

$m_n \geq 24 + 35 - 12 \geq 1$. Finally, the sequence of m_n is strictly increasing since $m_n \geq 24 + 35m_{n-1} - 12 = 35m_{n-1} + 12$ and the m_n are positive. Therefore, the m_n sequence leads to infinitely many positive integers with the required property.

An alternative way to find the recursive solution (1), is to write the equation $f(a, m) = 0$ in an equivalent form

$$3(m + 2a - 1)^2 - 11m^2 = 1.$$

The substitution $x = m + 2a - 1$ yields a Pell equation $3x^2 - 11m^2 = 1$. For an arbitrary Pell equation $cx^2 - dy^2 = 1$, the Pell resolvent is defined to be $u^2 - cdv^2 = 1$. Therefore, the Pell resolvent for $3x^2 - 11m^2 = 1$ is $u^2 - 33v^2 = 1$ with fundamental solution $(u_1, v_1) = (23, 4)$. Let $u_0 = 1$ and $v_0 = 0$. The general solution (u_n, v_n) for the Pell resolvent is recursively given for $n \geq 1$ by

$$\begin{aligned} u_{n+1} &= u_1u_n + cdv_1v_n = 23u_n + 132v_n \\ v_{n+1} &= v_1u_n + u_1v_n = 4u_n + 23v_n. \end{aligned} \tag{2}$$

Note that v_n is always an even integer and u_n is odd for each integer $n \geq 0$.

The general solution (x_n, m_n) for $3x^2 - 11m^2 = 1$ in terms of the solution of the resolvent is given by

$$\begin{aligned} x_n &= x_0u_n + dm_0v_n = 2u_n + 11v_n \\ m_n &= m_0u_n + cx_0v_n = u_n + 6v_n. \end{aligned} \tag{3}$$

Observe that x_n is an even integer and m_n is an odd integer for any $n \geq 0$. As a result, for x_n and m_n defined by (2), the equation $x_n = m_n + 2a - 1$ admits an integer solution a. Moreover, the recursive formulas (2) that define the u_n and v_n sequences can be used to derive the recursive formulas (1) for the a_n and m_n sequences.

We end by listing several examples.

n	m_n	a_n	Sum
0	1	1	$1^2 = 1^3$
1	47	22	$\underbrace{22^2 + 23^2 + \cdots + 68^2}_{47 \text{ terms}} = 47^3$
2	2161	989	$\underbrace{989^2 + 990^2 + \cdots + 3149^2}_{2161 \text{ terms}} = 2161^3$
3	99359	45450	$\underbrace{45450^2 + 45451^2 + \cdots + 144808^2}_{99359 \text{ terms}} = 99359^3$
4	4568353	2089689	$\underbrace{2089689^2 + 2089690^2 + \cdots + 6658041^2}_{4568353 \text{ terms}} = 4568353^3$
5	210044879	96080222	$\underbrace{96080222^2 + 96080223^2 + \cdots + 306125100^2}_{210044879 \text{ terms}} = 210044879^3$

OC420. General Tilly and the Duke of Wallenstein play "Divide and rule!" (Divide et impera!). To this end, they arrange N tin soldiers in M companies and command them by turns. Both of them must give a command and execute it in their turn.

Only two commands are possible: The command "Divide!" chooses one company and divides it into two companies, where the commander is free to choose their size, the only condition being that both companies must contain at least one tin soldier. On the other hand, the command "Rule!" removes exactly one tin soldier from each company.

The game is lost if in your turn you can't give a command without losing a company. Wallenstein starts to command.

(a) Can he force Tilly to lose if they start with 7 companies of 7 tin soldiers each?

(b) Who loses if they start with $M \geq 1$ companies consisting of $n_1 \geq 1$, $n_2 \geq 1$, $\dots$, $n_M \geq 1$ $(n_1 + n_2 + \cdots + n_M = N)$ tin soldiers?

Originally 2017 Germany Math Olympiad, 3rd Problem, Final Round.

We received 1 submission. We present the solution by Jeremy Mirmina.

We discuss the winning strategy of the game based on the parities (odd/even) of the number of tin soldiers N, the number of companies M, and the difference $I = N - M$.

First, notice the following. When Move 1 ("Divide!") is played N remains the same, M decreases by one, and I decreases by one and switches parity. When Move 2 ("Rule!") is played N decreases by M, M remains the same, and I decreases by M.

In the next table we summarise the changes in the parities of I, N, and M after Move 1 or Move 2 are played. We assume that the game did not end, and that Move 1 and Move 2 can be played.

				Move 1			Move 2	
I	N	M	new I	new N	new M	new I	new N	new M
odd	odd	even	even	odd	odd	odd	odd	even
odd	even	odd	even	even	even	even	odd	odd
even	even	even	odd	even	odd	even	even	even
even	odd	odd	odd	odd	even	odd	even	odd

Second, notice that if there is at least one company with exactly one soldier the two players can only use Move 1 and for exactly I times. This is because Move 1 will be applied $n_i - 1$ times to split a company with n_i soldiers into n_i companies, each with only one soldier.

Third, notice that the game ends when all companies have exactly one soldier.

Case 1. Assume I is odd, i.e. either N is even and M is odd or M is even and N is odd. Then the first player (Wallenstein) has a winning strategy.

Because $N \neq M$, he can choose a company with more than one soldier. He plays Move 1, and splits this company into one soldier and the rest. At this point, none of the two players can use Move 2, as it will make them lose. Move 1 is played exactly $I = N - M$ times until $I = 0$ and the game ends. The player who starts with I odd will continue to play with I odd and his opponent will play with I even. Eventually his opponent will receive the configuration with $I = 0$ and will lose. For this reason a player who moves with an even I, never wants to change it into odd on his opponent's turn.

Case 2. Assume I is even, and both N and M are odd. Then the second player (Tilly) has a winning strategy.

Based on the table above, the first player starts with an even I and regardless of his move he changes the parity of I to odd on his opponent's turn. Hence, the second player has always a winning strategy (see Case 1).

This answers part (a) of the problem, since $N = 7 \times 7 = 49$ is an odd number of soldiers and $M = 7$ is an odd number of companies. Wallenstein cannot force Tilly to loose, and Tilly, the second player, has a winning strategy.

Case 3. Assume I is even, and both N and M are even.

In this case, neither of the two players is interested in playing Move 1, which results in a winning configuration for the opponent. They play Move 2 for as long as they can. In fact, Move 2 can be played $m - 1$ times, where $m = \min\{n_i : i = 1, 2, \ldots, M\}$ is the size of the company with the lowest number of soldiers. If m is even then the first player (Wallenstein) has a winning strategy, and if m is odd then the second player (Tilly) has a winning strategy.

The conclusions of cases 1, 2, and 3 answer part (b).