TEACHING PROBLEMS

No.4

John McLoughlin

Handshakes with a Twist

The handshake problem is perhaps the most flexible problem in my experience working with K-12 teachers. The basic example of counting handshakes lends itself to modeling the results with people as the participants. For instance, four people can each shake hands with one another. The total of 6 handshakes is reached this way at any level. The case of five people can be a new problem, or an extension can be readily grasped using the idea of a fifth person arriving in the group, thus, adding 4 handshakes to give a total of 10 handshakes. Before proceeding further, it is assumed that the reader is convinced that a total of 10 handshakes would occur if all people in a group of five shook hands with one another.

Assuming that makes sense be aware of how you would obtain the result. There are at least three viable avenues including the idea of adding additional people to simpler cases, as outlined above. You may recognize that each person in the group must shake 4 hands suggesting that there may be $5 \times 4 = 20$ handshakes required. Of course, this is not 10, and so something must be awry. Yes, each handshake involves two people and so the number of handshakes in total would equal 20 divided by 2, or 10. Do you see that in general for n people there would be $n(n-1)/2$ handshakes required? Mathematically this is equivalent to ${}_nC_2$ or $\binom{n}{2}$ or "n choose 2", namely, the number of ways of selecting 2 people from a group of n people.

My experience suggests that a diagrammatic approach may offer another valuable way of representing the problem. You are invited to construct a diagram with five vertices, each representing a person. Joining all possible vertices with segments will illustrate that there are 10 possible ways of connecting two of the five people. These segments represent the handshakes. Following our discussion of some of the basic handshake counting principles, we are ready to tackle an unorthodox handshake problem. The use of a diagram may be helpful in considering the core problem of our discussion as stated below. Teachers may wish to have eight people represent the characters in the problem.

> Mr. and Mrs. Smith were at a party with three other married couples. Since some of the guests were not acquainted with one another, various handshakes took place. No one shook hands with his or her spouse, and of course, no one shook their own hand! After all of the introductions had been made, Mrs. Smith asked the other seven people how many hands each shook. Surprisingly, they all gave different answers. How many hands did Mr. Smith shake?

A detailed discussion of this problem appears in *Combinatorial Explorations*, a publication in the *ATOM Series* authored by Richard Hoshino and John Grant McLoughlin. "This problem is fascinating because it does not appear solvable. It is difficult to imagine that there is enough information here. However, we have all the information we need! Before reading any further, stop and attempt to solve this problem on your own."

Take some time and consider a diagram and/or a logical approach that makes it plausible to address the problem. It is not anticipated that you will necessarily solve the problem, as few of my students do in fact without some further guidance. In any case, play with the problem so that you may understand it better.

Here I will share two approaches, the first being a classroom approach and the second being the written approach.

Classroom Approach: Modeling the Problem

This approach requires 8 volunteers who are arranged into 4 couples, one of which is designated as Mr. and Mrs. Smith. Any names for the others are not a concern. It is easiest to place them in pairs square dancing, as if they are the four directions (N, S, E, W) on a compass.

Consider the important fact that Mrs. Smith received seven different answers to the number of hands shaken by the others in the group. In fact, there are only seven possible answers as the absence of a handshake with one's spouse limited the number of handshakes to a maximum of 6. That is, there must be people who accounted for each of 0, 1, 2, 3, 4, 5, and 6 handshakes. That is essential to getting started. Make sure that makes sense to you. The eighth person, Mrs. Smith, is not included on that list.

Here I ask one of the volunteers other than Mr. Smith to be the person shaking 6 hands. This person steps forward and proceeds to shake all possible hands (as in all people other than the spouse), thus, giving us a person with 6 handshakes. Do you see now that the spouse of this individual is the only person who could shake 0 hands? Hence, both members of that couple have completed their handshakes.

We proceed to identify another person other than Mr. Smith to be the person who will shake 5 hands. Observe that this person will have already counted 1 handshake and now can shake hands with Mr. and Mrs. Smith as well as with another couple. The spouse of the person shaking 5 hands will become the only person who can shake only 1 hand. Hence, these people have finished with shaking hands.

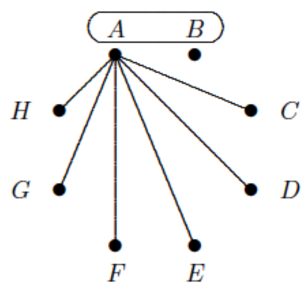
Finally, there is a volunteer who shakes more hands and becomes the person with 4 handshakes while having a spouse with only 2 handshakes. This leaves Mr. and Mrs. Smith each having shaken 3 hands. We are done with the handshakes and can definitively answer the question. How many hands did Mr. Smith shake? The answer is 3.

Written Approach: Using a Diagram

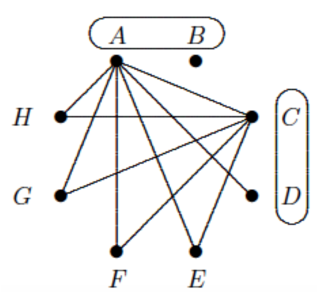
Let us model this problem graphically using a diagram. This can be thought of as a graph with 8 vertices labeled A, B, C, D, E, F, G, and H. Suppose that A is married to B, C is married to D, E is married to F, and G is married to H. Note that in the classroom model, Mr. and Mrs. Smith were positioned first. In this diagrammatic approach, their location will not be apparent until the completion of the process. That is, the coupling in pairs is required, but not the naming of any of the pairs.

Each vertex represents a person at the party. Two vertices will be joined if those two people shook hands. Again note that since no one shakes their own hand, or the hand of their spouse, a person can shake at most six hands. Thus, every person at the party shook at least 0 hands and at most 6 hands.

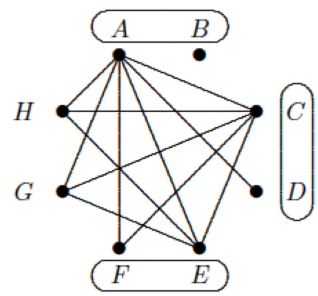
Consider the person who shook 6 hands. Let us assume this person is denoted as A. Hence, A shook hands with everyone at the party except for B. Represent this by drawing an edge from A to each of the other vertices in the graph, except for B. Thus, every person other than B has shaken at least one hand. Since someone at the party shook 0 hands, this implies that B must have been the person who shook 0 hands. This information is represented in the diagram below. In this diagram, the oval shape around A and B signifies that this couple has finished performing all of their handshakes.



Now consider the person who shook 5 hands. Assume this person is C. Then C must shake hands with each of E, F, G, and H. This shows that everyone (other than B and D) shook at least two hands. Therefore, it follows that D must have been the person who shook exactly 1 hand, as illustrated.



Now consider the person who shook 4 hands. Assume this person is E. Since E has already shaken two hands, E must shake hands with both G and H. Drawing edges from E to G and E to H we see that F must be the person who shook 2 hands. Further, both G and H have shaken at least three hands. The resulting diagram is shown:



Note that G and H are married and so they do not shake hands. Thus, both G and H shook three hands. We have now indicated all the handshakes that took place at this party. However, we need to identify Mr. and Mrs. Smith. If Mrs. Smith is any of A to F, two of the individuals would have replied to her question that they shook exactly three hands. That is a contradiction because all seven replies were different. Therefore, Mrs. Smith must be either G or H. Thus, Mrs. Smith shook three hands. Likewise, Mr. Smith as her spouse shook exactly 3 hands.

Questions and thoughts for consideration with the problem

i) Considering the classroom approach, why could we not have let Mr. Smith be the volunteer who shook extra hands?

Suppose that he began by shaking 6 hands. It would have then been impossible for any of the people other than Mrs. Smith to have had 0 handshakes, thus, violating the conditions of the problem. (Do you see that we needed someone else to shake no hands?) It is more difficult to consider other cases along the way, but you may wish to ponder those also.

ii) How does one know that Mrs. Smith had to shake an odd number of hands?

iii) Observe in the concluding diagram that all couples shook a total of 6 hands. Is this a coincidence or can you explain why this must be the case?

Some problems to try

Here are a few handshake questions to consider.

1. Suppose that twenty people attended a party, and everyone shook hands with exactly three guests. How many handshakes took place?

2. Mr. and Mrs. Smith were at a party with ten other married couples. Various handshakes took place. No one shook hands with their spouse, and of course, no one shook their own hand! After all the introductions had been made, Mrs. Smith asked the other people how many hands they shook. Surprisingly, they all gave a different answer. How many hands did Mr. Smith shake?

3. Everyone at a meeting shook hands with one another. Shortly after the meeting commenced, the chronically late character known as Tar D. arrived. Tar only managed to shake hands with some of the people present. In total, there were 59 handshakes. How many hands did Tar D. shake?

4. At a party attended by n people, various handshakes took place. Just for fun, each person shouted out the number of hands they shook. Explain why there must have been at least two people who shouted out the same number.

Reference

Hoshino, R. & Grant McLoughlin, J. (2005). *Combinatorial Explorations.* Ottawa: Canadian Mathematical Society.