# MathemAttic
# SOLUTIONS

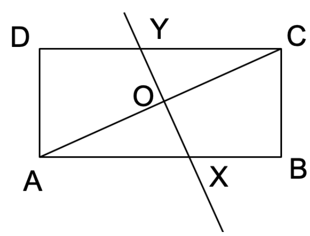
*Statements of the problems in this section originally appear in 2019: 45(2), p. 56–57.*

**MA6**. A rectangular sheet of paper is labelled $ABCD$, with $AB$ being one of the longer sides. The sheet is folded so that vertex $A$ is placed exactly on top of the opposite vertex $C$. The fold line is $XY$, where $X$ lies on $AB$ and $Y$ lies on $CD$. Prove that the triangle $CXY$ is isosceles.

*Originally problem B4 from 2018 UK Junior Mathematical Olympiad.*

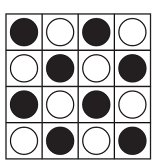
*We received 2 submissions both of which were correct and complete. We present the solution by Richard Hess, modified by the editor.*

Consider the below figure where $XY$ is the perpendicular bisector of $AC$ and $O$ is the center of the rectangle $ABCD$.




By symmetry, $OX = OY$. As $XY$ is perpendicular to $AC$, it follows that $CXY$ is an isosceles triangle where $CX = CY$.

**MA7**. Sixteen counters, which are black on one side and white on the other, are arranged in a 4 by 4 square. Initially all the counters are facing black side up. In one 'move', you must choose a 2 by 2 square within the square and turn all four counters over once. Describe a sequence of 'moves' of minimum length that finishes with the visible colours of the counters of the 4 by 4 square alternating (as shown in the diagram).



*Originally problem B6 from 2018 UK Junior Mathematical Olympiad.*

*We received 2 submissions of which 1 was correct and complete. We present the solution by Richard Hess, modified by the editor.*

Label the $1 \times 1$ squares left to right, bottom to top so that 1 refers to the bottom left square and 13 refers to the upper left square. First notice that the order in which the $2 \times 2$ squares are turned is inconsequential. The $1 \times 1$ squares in the bottom left and upper right begin as black, and must become white, hence the bottom left and upper right $2 \times 2$ squares must both by turned. This will be the first and second move. The squares 2,5,12, and 15 belong to exactly one $2 \times 2$ square, excluding the two squares turned in our first two moves. Since squares 2,5,12, and 15 are white after the second move, and must become black, we are forced to flip the middle bottom, middle right, middle left, and middle top $2 \times 2$ squares. After these next four moves, the checkerboard pattern is formed. It follows that 6 is the minimum number of moves.

**MA8**. I have two types of square tile. One type has a side length of 1 cm and the other has a side length of 2 cm. What is the smallest square that can be made with equal numbers of each type of tile?

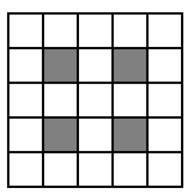
*Originally problem B5 from 2018 UK Junior Mathematical Olympiad.*

*We received 2 solutions. We present a solution based on the submission by Doddy Kastanya.*

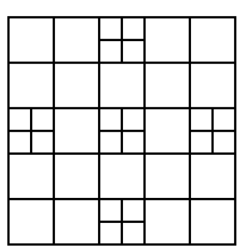
Suppose we have $N$ squares of each type tiling a square of side length $S$ (in cm$^2$). Then

$$S^2 = N \cdot 1 + N \cdot 4 = 5N.$$

The smallest $S$ that satisfies this equation is 5, which implies $N = 5$. However there is no possible arrangement of the tiles satisfying this, as can be seen from the figure below. Any $2 \times 2$ tile placed in the square covers exactly one of the four grey squares. Thus we cannot fit five $2 \times 2$ tiles into the $5 \times 5$ square.



The next possible $S$ satisfying the equation is 10, implying $N = 20$. A possible tiling is shown below.



Therefore the smallest square that can be made with equal numbers of each type of tiles has a side length of 10cm.

**MA9**. The letters $a, b, c, d, e$ and $f$ represent single digits and each letter represents a different digit. They satisfy the following equations:

$$a + b = d, \quad b + c = e, \quad d + e = f.$$

Find all possible solutions for the values of $a, b, c, d, e$ and $f$.

*Originally problem B8 from 2018 UK Junior Mathematical Olympiad.*

*We received two submissions both of which were correct and complete. We present the solution by Richard Hess, modified by the editor.*

Given $a + b = d$, $b + c = e$, $d + e = f$, we can combine equations to produce $f = a + c + 2b$. Given a solution $(a, b, c, d, e, f)$ that satisfies these equations, we can interchange $a$ and $d$ with $c$ and $e$, respectively, to find a second solution $(c, b, a, e, d, f)$. Thus, we need only search for solutions where $a < c$. As all digits are unique in the interval 1 to 9, the smallest $f$ can be is 7 $(a = 2, b = 1, c = 3)$. Below are the 8 solutions to this problem:

$$\begin{array}{ll} (2,1,4,3,5,8) & (4,1,2,5,3,8) \\ (2,1,5,3,6,9) & (5,1,2,6,3,9) \\ (1,2,4,3,6,9) & (4,2,1,6,3,9) \\ (1,3,2,4,5,9) & (2,3,1,5,4,9) \end{array}$$

**MA10**. An arithmetic and a geometric sequence, both consisting of only positive integral terms, share the same first two terms. Show that each term of the geometric sequence is also a term of the arithmetic sequence.

*Originally Problem* **J12**, *proposed by Colin Springer, from Mathematical Mayhem, 1 (1), p.19.*

*We received 5 submissions, of which all were correct and complete. We present the solution by Jacob Miles, modified by the editor.*

Let our arithmetic sequence be given by

$$a, a + d, a + 2d, \ldots$$

and our geometric sequence be given by

$$\alpha, \alpha r, \alpha r^2, \ldots$$

We show by proof by induction that each term $\alpha r^n$ of our geometric sequence is a term of our arithmetic sequence. We are given that $a = \alpha$ and $a + d = \alpha r$. Hence, the base cases of $n = 0, 1$ are satisfied. Let $n = k$. Assume $\alpha r^k$ is a term of our arithmetic sequence, i.e. $\alpha r^k = a + md$ for some $m \in \mathbb{N}$. We consider the case of $n = k + 1$ as follows

$$\begin{aligned} \alpha r^{k+1} &= \left(\alpha r^k\right) r \\ &= (a + md)r \\ &= ar + rmd. \end{aligned}$$

Given that $ar = a + d$, the above becomes

$$\begin{aligned} \alpha r^{k+1} &= a + d + rmd \\ &= a + (rm + 1)d. \end{aligned}$$

The above expression of $\alpha r^{k+1}$ is a member of the arithmetic sequence if $rm+1 \in \mathbb{N}$. As $m \in \mathbb{N}$ it suffices to prove that $r \in \mathbb{N}$. It is clear that $r > 0$ and that $r \in \mathbb{Q}$, for if either were not the case, we would have that $ar = a + d \notin \mathbb{N}$ which is a contradiction. We assume that $r \in \mathbb{Q}$ but $r \notin \mathbb{N}$. It follows that $r = \frac{x}{y}$ for some $x, y \in \mathbb{N}$ where $y \neq 1$. Without loss of generality, assume $\gcd(x, y) = 1$. As each term of our geometric sequence is a positive integer, we have that $ar^n \in \mathbb{N}$ for all $n \in \mathbb{N}$. Hence

$$ar^n = a\left(\frac{x}{y}\right)^n = \frac{ax^n}{y^n}.$$

We have that $y^n | ax^n$. However, since $\gcd(x, y) = 1$, it follows that $\gcd(x^n, y^n) = 1$. This implies $y^n | a$. However, as $y \neq 1$ we can choose a sufficiently large $n$ such that $y^n > a$ causing $y^n \nmid a$. This would imply that $ar^n \notin \mathbb{N}$. By proof by contradiction we conclude that $r \in \mathbb{N}$ and that every term in our geometric sequence is in our arithmetic sequence.