PROBLEM SOLVING VIGNETTES

No.7

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Careful Counting

This month we will look at problem B4 from the 2018 Canadian Open Mathematics Challenge administered by the CMS. You can check out past contests on the CMS webpage at `cms.math.ca/Competitions/COMC`.

> Determine the number of 5-tuples of integers $(x_1, x_2, x_3, x_4, x_5)$ so that
>
> (a) $x_i \geq i$ for $1 \leq i \leq 5$;
>
> (b) $\displaystyle\sum_{i=1}^{5} x_i = 25$.

Solution 1: We will look to see if we can find any patterns by considering possible solutions in an orderly manner. Suppose that we fix $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$. If we want the five numbers to add to 25, then $x_4 + x_5 = 25 - (1 + 2 + 3) = 19$. Recall that we also need $x_4 \geq 4$ and $x_5 \geq 5$. Putting this together, we get the following 11 5-tuples:

$$(1, 2, 3, 4, 15), (1, 2, 3, 5, 14), (1, 2, 3, 6, 13), \ldots, (1, 2, 3, 14, 5).$$

Next we will examine what happens when we allow x_3 to take on different values. We will keep $x_1 = 1$ and $x_2 = 2$ and let $x_3 = 4$. Using the same idea as in the first case we get 10 new 5-tuples:

$$(1, 2, 4, 4, 14), (1, 2, 4, 5, 13), (1, 2, 4, 6, 12), \ldots, (1, 2, 4, 13, 5).$$

Thus, if we fix $x_1 = 1$ and $x_2 = 2$, the total number of 5-tuples is

$$11 + 10 + 9 + \cdots + 1 = \sum_{i=1}^{11} i = 66.$$

Now, consider what happens if we fix $x_1 = 1$ and let $x_2 = 3$. If we go through the same process again, we get $10 + 9 + 8 + \cdots + 1 = 55$ more 5-tuples. So if we fix $x_1 = 1$ and let all other values vary we get

$$(11 + 10 + 9 + \cdots + 1) + (10 + 9 + 8 + \cdots + 1) + \cdots + (2 + 1) + 1 = \sum_{i=1}^{11} \sum_{j=1}^{i} j$$

5-tuples. Thus for the total problem, we will need to let x_1 vary. Letting $x_1 = 2$ and thinking through the process we get

$$(10+9+8+\cdots+1)+(9+8+7+\cdots+1)+\cdots+(2+1)+1=\sum_{i=1}^{10}\sum_{j=1}^{i} j$$

new 5-tuples. Hence looking at *all* possible solutions we must have

$$\begin{aligned}&[(11+10+9+\cdots+1)+\cdots+(2+1)+1]+\\ &\quad[(10+9+8+\cdots+1)+\cdots+(2+1)+1]+\cdots+[1]=\sum_{i=1}^{11}\sum_{j=1}^{i}\sum_{k=1}^{j} k. \qquad (1)\end{aligned}$$

In Vignette #5 [2019: 45(5), p. 236-240] we introduced and proved the following formulas:

$$1+2+3+\cdots+n=\sum_{i=1}^{n} i=\frac{n(n+1)}{2}, \qquad (2)$$

$$1^2+2^2+3^2+\cdots+n^2=\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}. \qquad (3)$$

We will add one more which will be of use to us in our solution:

$$1^3+2^3+3^3+\cdots+n^3=\sum_{i=1}^{n} i^3=\frac{n^2(n+1)^2}{2}. \qquad (4)$$

Enjoy practicing your induction by proving that the formula holds for all n.

Going back to (1), using (2), (3), and (4) we get

$$\begin{aligned}\sum_{i=1}^{11}\sum_{j=1}^{i}\sum_{k=1}^{j} k &= \sum_{i=1}^{11}\sum_{j=1}^{i}\frac{j(j+1)}{2}\\ &=\frac{1}{2}\sum_{i=1}^{11}\sum_{j=1}^{i}(j^2+j)\\ &=\frac{1}{2}\sum_{i=1}^{11}\left(\frac{i(i+1)(2i+1)}{6}+\frac{i(i+1)}{2}\right)\\ &=\frac{1}{6}\sum_{i=1}^{11}(i^3+3i^2+2i)\\ &=\frac{1}{6}\left(\frac{11^2\cdot 12^2}{4}+3\cdot\frac{11\cdot 12\cdot 23}{6}+2\cdot\frac{11\cdot 12}{2}\right)\\ &=1001.\end{aligned}$$

Therefore there are 1001 5-tuples that satisfy the conditions in the problem. □

Solution 2: We will look at the problem from another point of view. Suppose we wanted, for case of simplicity, to find all 3-tuples of non-negative integers (x_1, x_2, x_3) such that $x_1 + x_2 + x_3 = 5$. This is a simplification of our problem by considering only 3 numbers, having a smaller sum and letting them all be any non-negative integer. For this problem we could list out all the possibilities or count them by carefully looking at cases.

Case 1: Two of the numbers are the same (there is no way they can all be the same). There are three ways that this can happen: $(0, 0, 5)$, $(1, 1, 3)$, and $(2, 2, 1)$. For each of these cases there are $\frac{3!}{2!} = 3$ ways to arrange the numbers giving $3 \cdot 3 = 9$ 3-tuples: $(0, 0, 5)$, $(0, 5, 0)$, $(5, 0, 0)$, $(1, 1, 3)$ $(1, 3, 1)$, $(3, 1, 1)$, $(2, 2, 1)$, $(2, 1, 2)$, and $(1, 2, 2)$.

Case 2: None of the numbers are the same. There are only two ways that this can happen: $(0, 1, 4)$ and $(0, 2, 3)$. For each of these cases there are $3! = 6$ ways to arrange the numbers giving $2 \cdot 6 = 12$ more 3-tuples: $(0, 1, 4)$, $(0, 4, 1)$, $(1, 0, 4)$, $(1, 4, 0)$, $(4, 0, 1)$, $(4, 1, 0)$, $(0, 2, 3)$, $(0, 3, 2)$, $(2, 0, 3)$, $(2, 3, 0)$, $(3, 0, 2)$, and $(3, 2, 0)$.

Therefore there are $9 + 12 = 21$ 3-tuples in total. We can use this method on our problem, but there will be many more cases to look at. You may (or may not!) want to see if you can identify all cases and get the correct total of 1001.

Still looking at the simplified problem, suppose we represent any particular 3-tuple with a collection of *stars and bars* (the name usually associated with this method). We will use five stars, since the total is 5 and two bars to separate them into three groups. Thus the 3-tuple $(2, 1, 2)$ would be represented by $**\,|\,*\,|\,**$. All stars to the left of the first bar represent x_1, the stars between the bars represent x_2 and the stars to the right of the second bar represents x_3. Similarly $*||****$ would represent $(1, 0, 4)$ and $*****||$ would represent $(5, 0, 0)$.

Every 3-tuple can be represented by a unique permutation of 5 stars and 2 bars. Similarly, every permutation of 5 stars and 2 bars represents a unique 3-tuple. There is a one-to-one correspondence between the 3-tuples and the permutations of 5 stars and 2 bars. Since the total number of permutations of 5 stars and 2 bars is $\frac{7!}{5!2!} = 21$, we solved our simplified problem in a much more efficient manner.

If we return to the original problem, all permutations of 25 stars and 4 bars would give all possible 5-tuples of non-negative integers that sum to 25. This is not quite what we are after, **but** if we let $x_i = i + y_i$, for $1 \leq i \leq 5$ then $(y_1, y_2, y_3, y_4, y_5)$ is a 5-tuple of non-negative integers that add to $25 - (1 + 2 + 3 + 4 + 5) = 10$ and there is a one-to-one correspondence between the 5-tuples $(y_1, y_2, y_3, y_4, y_5)$ and the 5-tuples that we are after. That is, for example, since $(3, 1, 4, 2, 0)$ is a collection of y_is, then $(3 + 1, 1 + 2, 4 + 3, 2 + 4, 0 + 5) = (4, 3, 7, 6, 5)$ is an allowed solution to the original problem. The number of possible 5-tuples $(y_1, y_2, y_3, y_4, y_5)$ is the same as the number of permutations of 10 stars and 4 bars or

$$\frac{14!}{10!4!} = 1001.$$

□

The stars and bars method yields a solution much quicker. We can summarize it as follows: the number of distinct n-tuples of non-negative integers whose sum is s is

$$\frac{(s+n-1)!}{s!(n-1)!}.$$

This can be generalized to the following statement: the number of distinct n-tuples of integers, $(x_1, x_2, \ldots, x_n)$ whose sum is s, where $x_i \leq m_i$ for $1 \leq i \leq n$ is

$$\frac{(s+(n-1)-\sum_{i=1}^{n} m_i)!}{(s-\sum_{i=1}^{n} m_i)!(n-1)!}.$$

We will explore other counting techniques in future columns.