FOCUS ON...

No. 37

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Geometry with Complex Numbers (II)

Introduction

In this second part, we continue to present various interventions of the complex numbers in geometry problems. We begin with regular polygons, an obvious domain of application. Then, we will consider similarities, either direct or opposite, as they can be simply represented using complex numbers, and we conclude with a look at areas.

Complex numbers and regular polygons

It is well-known that the n-th roots of a nonzero complex number are the affixes of the vertices of a regular n-gon. In particular, the n-th roots of unity $\exp(2k\pi i/n)$, $k = 0, 1, \ldots, n-1$, correspond to an n-gon inscribed in the unit circle Γ, with centre O and radius 1. Here is a first illustration, a problem proposed in the December 2017 issue of *Mathematics Magazine*:

> Let n be an integer, $n \geq 2$. Let $A_1A_2A_3 \cdots A_{2n+1}$ be a regular polygon with $2n+1$ sides. Let P be the intersection of the segments $\overline{A_2A_{n+2}}$ and $\overline{A_3A_{n+3}}$. Prove that
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> $$(A_1P)^2 = (A_2A_3)^2 + (A_3P)^2.$$

We may suppose that for $k = 1, 2, \ldots, 2n+1$, A_k is the point with complex affix w^{k-1} where $w = \exp\left(\frac{2\pi i}{2n+1}\right)$. Let p be the affix of P. We readily obtain

$$(A_2A_3)^2 = |w^2 - w|^2 = |w-1|^2 = (w-1)(\overline{w}-1) = 2 - w - \overline{w} = 2 - w - w^{2n}.$$

and

$$\begin{aligned}(A_1P)^2 - (A_3P)^2 &= |p-1|^2 - |p-w^2|^2 \\ &= (p-1)(\overline{p}-1) - (p-w^2)(\overline{p}-w^{2n-1}) \\ &= pw^{2n-1} - p + \overline{p}w^2 - \overline{p}.\end{aligned}$$

Now, from the equations of the lines A_2A_{n+2} and A_3A_{n+3}, we deduce that $pw^n + \overline{p}w = 1 + w^{n+1}$ and $pw^{n-1} + \overline{p}w^2 = 1 + w^{n+1}$. It follows that

$$pw^{2n-1} - p = w^n(pw^{n-1} + \overline{p}w^2) - w^{n+1}(pw^n + \overline{p}w) = w^n + 1 - (w^{n+1} + w)$$

$$\overline{p}w^2 - \overline{p} = (pw^{n-1} + \overline{p}w^2) - \frac{1}{w}(pw^n + \overline{p}w) = 1 + w^{n+1} - (w^{2n} + w^n)$$

and by addition,

$$(A_1P)^2 - (A_3P)^2 = 2 - w - w^{2n} = (A_2A_3)^2.$$

As a second example, we prove a vectorial result about the projections of a point on the sidelines of a regular polygon:

> Let the consecutive vertices of a regular n-gon be denoted $A_0, \ldots, A_{n-1}$, in order, and let $A_n = A_0$. Let B_k be the projection of a point M onto the line A_kA_{k+1}. Show that
>
> $$\sum_{k=0}^{n-1} \overrightarrow{MB_k} = \frac{n}{2}\overrightarrow{MO}.$$

Again, we suppose that the affix of A_k is w^k with $w = \exp(2\pi i/n)$. Let m be the affix of M and let C_k be the midpoint of A_kA_{k+1}.

We remark that $\sum_{k=0}^{n-1} \overrightarrow{MC_k} = n\overrightarrow{MO}$; for example because

$$\sum_{k=0}^{n-1}\left(\frac{w^k + w^{k+1}}{2} - m\right) = \left(\sum_{k=0}^{n-1} w^k\right) - nm = \frac{1-w^n}{1-w} - nm = -nm$$

and

$$\overrightarrow{C_kB_k} = \frac{(\overrightarrow{OM} \cdot \overrightarrow{A_kA_{k+1}})}{\|\overrightarrow{A_kA_{k+1}}\|^2}\,\overrightarrow{A_kA_{k+1}}.$$

With the notations $m_k, \delta_k = w^{k+1} - w^k$ for the complex affixes of $\overrightarrow{C_kB_k}, \overrightarrow{A_kA_{k+1}}$, respectively, the latter gives

$$m_k = (\mathrm{Re}(m\overline{\delta_k}))\frac{\delta_k}{\delta_k\overline{\delta_k}} = \frac{1}{2}\cdot\frac{(m\overline{\delta_k} + \overline{m}\delta_k)}{\overline{\delta_k}} = \frac{1}{2}(m - \overline{m}w^{2k+1}).$$

Since

$$\sum_{k=0}^{n-1} w^{2k+1} = w \cdot \frac{1-(w^n)^2}{1-w^2} = 0,$$

we obtain $\sum_{k=0}^{n-1} m_k = \frac{n}{2}\cdot m$, that is,

$$\sum_{k=0}^{n-1} \overrightarrow{C_kB_k} = \frac{n}{2}\,\overrightarrow{OM}.$$

The desired result then follows from

$$\sum_{k=0}^{n-1}\overrightarrow{MB_k} = \sum_{k=0}^{n-1}\overrightarrow{MC_k} + \sum_{k=0}^{n-1}\overrightarrow{C_kB_k} = n\overrightarrow{MO} + \frac{n}{2}\,\overrightarrow{OM}.$$

(The established result will be used later, in the paragraph devoted to areas.)

Complex numbers and similarities

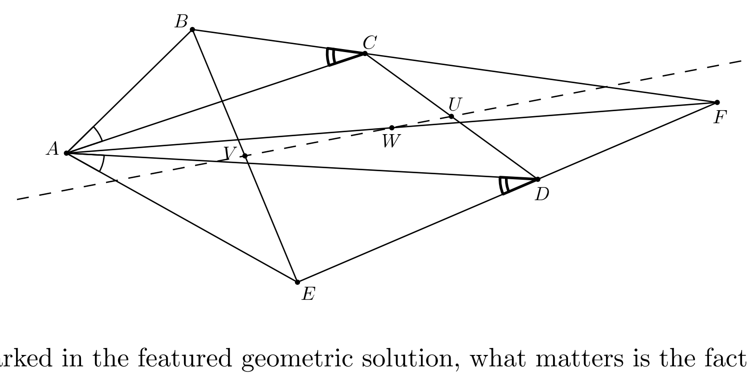
A similarity with factor $k > 0$ is a transformation of the plane such that the distance between the images of M, N is kMN for any points M, N of the plane. The similarities with factor 1 are the isometries while those of factor $k \neq 1$ are of the form $\mathcal{H} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{H}$ where $\mathcal{H}$ is a homothety with factor k and centre Ω and $\mathcal{I}$ is an isometry such that $\mathcal{I}(\Omega) = \Omega$. The similarity is direct or opposite according as $\mathcal{I}$ is a displacement or a reflection in a line (that is, according as it preserves orientation or not). Embedding all this in the complex plane, it can be showed that a direct (resp. opposite) similarity transforms a point M with affix m into the point M' with affix $m' = am + b$ (resp. $m' = a\overline{m} + b$) for some complex numbers $a \neq 0, b$ independent of m. To illustrate the power of this complex representation, we consider two examples, the first one being adapted from an exercise set at the French final high-school exam long ago:

> Let OAB be a triangle and suppose that points P and Q are such that the triangles $OA'B', AB'P, BQA'$ are directly similar to the triangles OAB, ABO, BOA, respectively. Show that O is the midpoint of PQ.

Taking O as the origin and using the corresponding lower-case letter for the affix of any other point, the hypotheses imply that for some complex numbers u, u_1, u_2, we have $a' = ua,\ b' = ub, b' = u_1 b + p,\ a = u_1 a + p,\ a' = u_2 a + q,\ b = u_2 b + q$ (for example, the latter because some direct similarity transforms A into A', B into B and O into Q). The elimination of u_1 and u_2 leads to $p(a-b) = ab(u-1)$ and $q(b-a) = ab(u-1)$. The desired relation $p + q = 0$ follows.

As a second example, we offer a variant of solution to problem **3401** [2009 : 42, 44 ; 2010 : 49]:

> Let $ABCDE$ be a convex pentagon such that $\angle BAC = \angle EAD$ and $\angle BCA = \angle EDA$, and let the lines CB and DE intersect in the point F. Prove that the midpoints of CD, BE, and AF are collinear.



As remarked in the featured geometric solution, what matters is the fact that the triangles ABC and AED are oppositely similar. Therefore, taking the point A as the origin, there exists a complex number ω such that $e = \omega\overline{b}$ and $d = \omega\overline{c}$. Let U, V, W be the midpoints of CD, BE, AF, respectively. Their affixes are $u = \frac{1}{2}(c + \omega\overline{c})$, $v = \frac{1}{2}(b + \omega\overline{b})$, and $w = \frac{1}{2} \cdot f$.

The equations of the lines BC and DE are readily obtained:

$$\overline{z}(c-b) - z(\overline{c}-\overline{b}) = (\overline{b}c - b\overline{c})$$

and

$$\omega\overline{z}(\overline{c}-\overline{b}) - \overline{\omega}z(c-b) = |\omega|^2(b\overline{c}-\overline{b}c).$$

This said, U, V, W are collinear if and only if

$$(\overline{u}-\overline{w})(v-w) = (u-w)(\overline{v}-\overline{w})$$

or

$$(\overline{c}+\overline{\omega}c-\overline{f})(b+\omega\overline{b}-f) = (c+\omega\overline{c}-f)(\overline{b}+\overline{\omega}b-\overline{f}).$$

Expanding and arranging, this condition can be written as

$$f(\overline{b}-\overline{c}) - \overline{f}(b-c) + b\overline{c} - \overline{b}c = \omega\overline{f}(\overline{b}-\overline{c}) - \overline{\omega}f(b-c) + |\omega|^2(b\overline{c}-\overline{b}c).$$

This certainly holds since both sides are 0 (the left-hand side because F is on BC and the right one because F is on DE). The conclusion follows.

Complex numbers and area

We start with the following known expression of the area $[ABCD]$ of a quadrilateral $ABCD$:

$$[ABCD] = \pm\frac{1}{2}AC \cdot BD \cdot \sin(\angle(\overrightarrow{AC}, \overrightarrow{BD}))$$

where the sign is + if and only if the quadrilateral is positively oriented, and $\angle(\overrightarrow{AC}, \overrightarrow{BD})$ is the directed angle from $\overrightarrow{AC}$ to $\overrightarrow{BD}$. If a, b, c, d are the affixes of the vertices, we obtain the following formula:

$$[ABCD] = \pm\frac{1}{2}\text{Im}((d-b)(\overline{c}-\overline{a})),$$

of which, for convenience, we repeat the proof.

Let $\alpha = \arg(d-b)$ and $\beta = \arg(c-a)$. Then, $\sin(\angle(\overrightarrow{AC}, \overrightarrow{BD})) = \sin(\alpha-\beta)$, hence

$$\begin{aligned}[ABCD] &= \pm\frac{1}{2}\,|c-a| \cdot |d-b| \sin(\alpha-\beta) \\ &= \pm\frac{1}{2}\,|c-a| \cdot |d-b| \text{Im}(e^{i(\alpha-\beta)}) \\ &= \pm\frac{1}{2}\,\text{Im}(|d-b|e^{i\alpha} \cdot |c-a|e^{-i\beta}),\end{aligned}$$

that is,

$$[ABCD] = \pm\frac{1}{2}\,\text{Im}\left((d-b)(\overline{c-a})\right).$$

Note that taking $d = c$, we obtain a formula for the area of ΔABC:

$$[ABC] = \pm\frac{1}{2}\,\text{Im}\left((c-b)(\overline{c}-\overline{a})\right).$$

We will see these results at work in two examples. First, we again consider the projections B_k of a point M on the sides of a regular n-gon $A_0A_1 \ldots A_{n-1}$ (with $A_0 = A_n$) and we suppose that the point B_k lies on the segment A_kA_{k+1} for $k = 0, 1, \ldots, n-1$ (see the second problem of the second paragraph). We will show that $S_1 = S_2$ where

$$S_1 = \sum_{k=0}^{n-1} [MA_kB_k] \quad \text{and} \quad S_2 = \sum_{k=0}^{n-1} [MA_{k+1}B_k].$$

We use the notations used earlier and denote by b_k and c_k the affixes of B_k and C_k. Observing that ΔMA_kB_k and $\Delta MA_{k+1}B_k$ have opposite orientations, proving that $S_1 - S_2 = 0$ amounts to proving that

$$\sum_{k=0}^{n-1} \mathrm{Im}[(m - b_k)(\overline{w_k} - \overline{b_k}) + (m - b_k)(\overline{w_{k+1}} - \overline{b_k})] = 0.$$

Now, we have

$$\begin{aligned}(m - b_k)(\overline{w_k} - \overline{b_k}) + (m - b_k)(\overline{w_{k+1}} - \overline{b_k}) &= 2(m - b_k)(\overline{c_k} - \overline{b_k}) \\ &= 2(m - b_k)(\overline{c_k} - \overline{m}) + 2|m - b_k|^2\end{aligned}$$

and we remark that $(m - b_k)\overline{c_k}$ is a real number (since OC_k is parallel to MB_k). It follows that

$$\sum_{k=0}^{n-1} \mathrm{Im}[(m - b_k)(\overline{w_k} - \overline{b_k}) + (m - b_k)(\overline{w_{k+1}} - \overline{b_k})] = -2 \cdot \mathrm{Im}\left(\overline{m} \sum_{k=0}^{n-1} (m - b_k)\right).$$

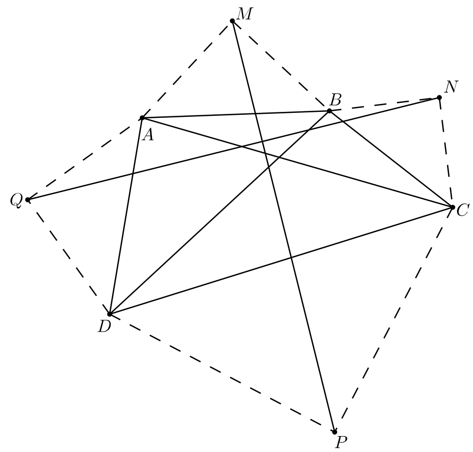
But we have $\sum\limits_{k=0}^{n-1} (m - b_k) = \frac{n}{2} \cdot m$ (from $\sum\limits_{k=0}^{n-1} \overrightarrow{MB_k} = \frac{n}{2}\overrightarrow{MO}$ proved earlier) so that

$$-2 \cdot \mathrm{Im}(\overline{m} \sum_{k=0}^{n-1} (m - b_k)) = -n\mathrm{Im}(m\overline{m}) = 0$$

and consequently $S_1 - S_2 = 0$.

Our second example is adapted from a problem set in the *Mathematical Gazette* in 2017:

> Squares are described externally on the sides of a convex quadrilateral $ABCD$. Prove that the line segments joining the centres of opposite squares are perpendicular and that the length of each line segment is $\sqrt{2S + \frac{1}{2}(x^2 + y^2)}$, where S is the area of $ABCD$ and $x = AC$, $y = BD$.



Assuming that $ABCD$ is clockwise oriented and with obvious notations, we have $b-m = i(a-m)$, hence $(1-i)m = b-ia$. Similarly, $(1-i)n = c-ib$, $(1-i)p = d-ic$ and $(1-i)q = a-id$. It follows that $q-n = -i(p-m)$ so that $NQ = PM$ and $NQ \perp PM$. Furthermore,

$$2MP^2 = |(1-i)(p-m)|^2 = [d-b-i(c-a)] \cdot [\overline{d} - \overline{b} + i(\overline{c} - \overline{a})] = \alpha + i\beta$$

where $\alpha = |d-b|^2 + |c-a|^2 = x^2 + y^2$ and

$$\beta = (d-b)(\overline{c} - \overline{a}) - (\overline{d} - \overline{b})(c-a) = 2i\mathrm{Im}[((d-b)(\overline{c} - \overline{a})] = -4i[ABCD].$$

Finally, $2MP^2 = x^2 + y^2 + 4S$, as desired.

As usual, we end the number with a couple of exercises.

Exercises

1. Let C be a point distinct from the vertices of a triangle OAB. Suppose that ΔOCD and ΔCAE are directly similar to ΔOAB. Prove that $CDBE$ is a parallelogram.

2. Use complex numbers to solve problem **3898**: On the extension of the side AB of the regular pentagon $ABCDE$, let the points F and G be placed in the order F, A, B, G so that $AG = BF = AC$. Compare the area of triangle FGD to the area of pentagon $ABCDE$.