SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2018: 44(8), p. 340–343; 44(9), p.387–390; 44(10): 423–426.

4376. *Proposed by Marius Drăgan and Neculai Stanciu.*

Let A and B be two matrices in $M_n(\mathbb{C})$ such that $AB = -BA$. Prove that

$$\det(A^4 + A^2B^2 + 2A^2 + I_n) \geq 0.$$

We received 6 solutions and will feature just one of them here. We present the solution by Missouri State University Problem Solving Group.

The result is false as stated. Let

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where $\alpha = (1+i)/\sqrt{2}$. Then $AB = -BA$, but

$$\det\left(A^4 + A^2B^2 + 2A^2 + I_n\right) = \left(1+\alpha^2\right)^4 = -4.$$

We assume that the intended hypothesis was $A, B \in M_n(\mathbb{R})$. Note that

$$\begin{aligned}
&\left(I_n + A^2 + AB\right)\left(I_n + A^2 + BA\right) \\
&= I_n + A^2 + BA + A^2 + A^4 + A^2BA + AB + ABA^2 + ABBA \\
&= I_n + A^2 - AB + A^2 + A^4 + A^2BA + AB - A^2BA - ABAB \\
&= I_n + A^2 - AB + A^2 + A^4 + A^2BA + AB - A^2BA + A^2B^2 \\
&= A^4 + A^2B^2 + 2A^2 + I_n.
\end{aligned}$$

Now

$$\begin{aligned}
\det\left(I_n + A^2 + AB\right) &= \det\left(I_n + A\left(A+B\right)\right) \\
&= \det\left(I_n + \left(A+B\right)A\right) \\
&= \det\left(I_n + A^2 + BA\right),
\end{aligned}$$

using Sylvester's Theorem that $\det(I_n + XY) = \det(I_n + YX)$. Finally, we have

$$\begin{aligned}
\det\left(A^4 + A^2B^2 + 2A^2 + I_n\right) &= \det\left(\left(I_n + A^2 + AB\right)\left(I_n + A^2 + BA\right)\right) \\
&= \det\left(I_n + A^2 + AB\right)\det\left(I_n + A^2 + BA\right) \\
&= \left(\det\left(I_n + A^2 + AB\right)\right)^2 \\
&\geq 0.
\end{aligned}$$

4377. *Proposed by Tidor Vlad Pricopie and Leonard Giugiuc.*

Let $x \geq y \geq z > 0$ such that $x + y + z + xy + xz + yz = 1 + xyz$. Find $\min x$.

We received 7 correct solutions and 1 incorrect submission. We feature the solution based on the approach of Ramanujan Srihari and the collaboration between Leonard Giugiuc and Tidor Vlad Pricopie, done independently.

We first show that $xy+xz+yz \neq 1$. Otherwise, we would have that $x+y+z = xyz$ and

$$z = z(xy + xz + yz) = x + y + z + z^2(x + y),$$

leading to

$$0 = (x + y)(1 + z^2),$$

which contradicts the hypotheses.

Let $x = \tan u$, $y = \tan v$, $z = \tan w$ where $\pi/2 > u \geq v \geq w > 0$. Then

$$\tan(u + v + w) = \frac{x + y + z - xyz}{1 - xy - xz - yz} = 1.$$

Therefore

$$u + v + w = \pi/4 \quad \text{or} \quad u + v + w = 5\pi/4.$$

In the latter case, we would have $u > 5\pi/12$ and $x > 1$.

When $u + v + w = \pi/4$, then $u \geq \pi/12$. From the formula for $\tan 3\theta$, we see that $\tan \pi/12$ satisfies the equation

$$0 = t^3 - 3t^2 - 3t + 1 = (t + 1)(t^2 - 4t + 1),$$

so that $x \geq 2 - \sqrt{3}$. However, it is possible that $u = v = w = \pi/12$, so that the equation is satisfied by $x = y = z = 2 - \sqrt{3}$.

Therefore, for all positive solutions of the equation, $x \geq 2 - \sqrt{3}$, and the lower bound is attained.

4378. *Proposed by Tarit Goswami.*

Find all k such that the following limit exists

$$\lim_{n\to\infty} \{k \cdot F_{n+1} - \sum_{i=0}^{n} \tau^i\},$$

where F_n is the n^{th} Fibonacci number and τ is the golden ratio.

Five correct solutions and three incorrect or incomplete solutions were received to the problem when the brackets are used to simply display the summand. The first solution by Sushanth Sathish Kumar solves this problem. However, the Problem Solving Group at Missouri State University took the brackets to refer to the fractional part. Theirs is the second solution below.

Solution 1, by Sushanth Sathish Kumar.

Using Binet's formula for F_{n+1} and the relation $1-\tau=-1/\tau$, we have that

$$\begin{aligned}kF_{n+1}-\sum_{i=0}^{n}\tau^i &= \frac{k}{\sqrt{5}}(\tau^{n+1}-(1-\tau)^{n+1})-\frac{\tau^{n+1}-1}{\tau-1}\\ &= \frac{k}{\sqrt{5}}(\tau^{n+1}-(1-\tau)^{n+1})-\tau^{n+2}+\tau\\ &= \tau+\left(\frac{k}{\sqrt{5}}-\tau\right)\tau^{n+1}-\frac{k}{\sqrt{5}}(1-\tau)^{n+1}.\end{aligned}$$

Since $\lim_{n\to\infty}\tau^{n+1}=\infty$ and $\lim_{n\to\infty}(1-\tau)^{n+1}=0$, the limit can exist if and only if $k=\sqrt{5}\tau=(5+\sqrt{5})/2$, in which case the limit is τ.

Solution 2, by Missouri State University Problem Solving Group.

Let

$$x_n = kF_{n+1}-\sum_{i=0}^{n}\tau^i$$

and let $\{x_n\}=x_n-\lfloor x_n\rfloor$ be the fractional part of x_n. We prove that the limit of $\{x_n\}$ exists if and only if $k=u+v\tau$ for some integers u and v.

Suppose first that $k=u+v\tau$. Since $\tau^n=\tau F_n+F_{n-1}$ for each positive integer n,

$$x_n = kF_{n+1}-(\tau^{n+2}-\tau)=(k-1)F_{n+1}-\tau F_{n+2}+\tau. \tag{1}$$

By Binet's formula, $\lim_{n\to\infty}\tau F_n-F_{n+1}=0$, so that

$$\begin{aligned}x_n &= (k-1)F_{n+1}-\tau F_{n+2}+\tau\\ &= (u-1)F_{n+1}+v\tau F_{n+1}-\tau F_{n+2}+\tau\\ &= (u-1)F_{n+1}+vF_{n+2}-F_{n+3}+1+(\tau-1+\epsilon_n)\end{aligned}$$

where $\lim_{n\to\infty}\epsilon_n=0$. Hence, for sufficiently large n, $\{x_n\}=\tau-1+\epsilon_n$ and so $\lim_{n\to\infty}\{x_n\}=\tau-1$.

Now, suppose that $\lim\{x_n\}=L$ and let

$$y_n=\lfloor x_n\rfloor-1=x_n-\{x_n\}-1.$$

Then, for $n\geq 3$, from (1), we have that

$$\begin{aligned}y_n-y_{n-1}-y_{n-2} &= (x_n-x_{n-1}-x_{n-2})-(\{x_n\}-\{x_{n-1}\})-\{x_{n-2}\})+1\\ &= -\tau+(\{x_{n-1}\}+\{x_{n-2}\}-\{x_n\})+1.\end{aligned}$$

Since $\{y_n-y_{n-1}-y_{n-2}\}$ is a convergent sequence of integers with limit $L-(\tau-1)$, it is eventually constant. Since $0\leq L\leq 1$ and $0<\tau-1<0$, the limit must be 0 and so $y_n=y_{n-1}+y_{n-2}$ for $n\geq N$. Therefore, for $n\geq N$,

$$y_n = aF_{n+1-N}+bF_{n-N}$$

for $n \geq N$, where $a = y_N$ and $b = y_{N-1}$.

From (1),

$$\begin{aligned} k-1 &= \frac{x_n + \tau F_{n+2} - \tau}{F_{n+1}} \\ &= \frac{y_n + \{x_n\} + 1 + \tau F_{n+2} - \tau}{F_{n+1}} \\ &= a\frac{F_{n+1-N}}{F_{n+1}} + b\frac{F_{n-N}}{F_{n+1}} + \tau\frac{F_{n+2}}{F_{n+1}} + \frac{\{x_n\} - (\tau - 1)}{F_{n+1}}. \end{aligned}$$

Let n tend to infinity. Then

$$\begin{aligned} k-1 &= a\tau^{-N} + b\tau^{-(N+1)} + \tau^2 \\ &= a(\tau-1)^N + b(\tau-1)^{N+1} + \tau^2, \end{aligned}$$

so that $k = u + v\tau$ for some integers u and v.

4379. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Let triangle ABC share its vertices with three vertices of a regular heptagon; in particular, let B coincide with vertex 1, C with vertex 2, and A with vertex 4. Let I be the incenter and let G be the centroid of ABC, respectively. Suppose BI intersects AC in D and CI intersects AB in E. Show that the points D, G and E are collinear.

We received 5 solutions. Presented is the one by Andrea Fanchini, lightly edited.

We use barycentric coordinates with reference to the triangle ABC, with side lengths a, b, and c. The vertices of the triangle have coordinates $A(a:0:0)$, $B(0:b:0)$ and $C(0:0:c)$, the incentre has coordinates $I(a:b:c)$ and the centroid $G(1:1:1)$. Using that D is on both lines BI and AC, we obtain the coordinates $D(a:0:c)$. Similarly we get $E(a:b:0)$. The points G, D and E are collinear if and only if

$$\begin{vmatrix} 1 & 1 & 1 \\ a & 0 & c \\ a & b & 0 \end{vmatrix} = ac - bc + ab = 0.$$

Now we know that the side lengths of the heptagonal triangle satisfy the relations

$$bc = c^2 - a^2, \quad ac = b^2 - a^2, \quad ab = c^2 - b^2.$$

After substituting these relations, we are done.

4380. *Proposed by George Apostolopoulos.*

Let a, b and c be the side lengths of a triangle ABC with inradius r and circumradius R. Prove that

$$a^2 \tan \frac{A}{2} + b^2 \tan \frac{B}{2} + c^2 \tan \frac{C}{2} \leq \frac{3\sqrt{3}R^3(R-r)}{2r^2}.$$

We received 12 submissions, including the one from the proposer, all correct. We present two solutions, the second one of which gives a sharper inequality.

Solution 1, by Kee-Wai Lau.

Let S denote the semiperimeter of triangle ABC. The following identities and inequalities are all well known:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \tag{1}$$

$$\sin A \cos A + \sin B \cos B + \sin C \cos C = \frac{rS}{R^2} \tag{2}$$

$$R \geq 2r \quad \textit{(Euler's Inequality)} \tag{3}$$

$$s \leq \frac{3\sqrt{3}R}{2} \tag{4}$$

By (1) we have

$$\begin{aligned} a^2 \tan \frac{A}{2} &= 4R^2(\sin^2 A)(\tan \frac{A}{2}) \\ &= 4R^2(\sin A)(2 \sin \frac{A}{2} \cos \frac{A}{2})(\tan \frac{A}{2}) \\ &= 8R^2(\sin A)(\sin^2 \frac{A}{2}) \\ &= 4R^2(\sin A)(1 - \cos A). \end{aligned}$$

Similarly, $b^2 \tan \frac{B}{2} = 4R^2(\sin B)(1 - \cos B)$ and $c^2 \tan \frac{C}{2} = 4R^2(\sin C)(1 - \cos C)$. Hence, by (1), (2), (3), and (4) we have

$$\begin{aligned} \sum_{cyc} a^2 \tan \frac{A}{2} = 4R^2\Big(\sum_{cyc} \sin A - \sum_{cyc} \sin A \cos A\Big) &= 4R^2\Big(\frac{a+b+c}{2R} - \frac{rs}{R^2}\Big) \\ &= 4s(R-r) \\ &\leq 6\sqrt{3}R(R-r) \\ &= \frac{3\sqrt{3}R(2r)^2(R-r)}{2r^2} \\ &\leq \frac{3\sqrt{3}R^3(R-r)}{2r^2} \end{aligned}$$

and we are done.

Solution 2, by Arkady Alt.

We prove the inequality that

$$\sum_{cyc} a^2 \tan\frac{A}{2} \le 6\sqrt{3}R(R-r)$$

which is sharper than the proposed result since

$$6\sqrt{3}R(R-r) \le \frac{3\sqrt{3}R^3(R-r)}{2r^2} \iff 2r \le R$$

which is Euler's Inequality.

Using the known results that

$$\tan\frac{A}{2} = \frac{r}{s-a}, \quad \sum_{cyc}\frac{a}{s-a} = \frac{4R-2r}{r} \quad \text{and} \quad s \le \frac{3\sqrt{3}R}{2},$$

we obtain

$$\begin{aligned}
&\sum_{cyc} a^2 \tan\frac{A}{2} \le 6\sqrt{3}R(R-r) \iff \\
&\sum_{cyc}\frac{a^2}{s-a} \le \frac{6\sqrt{3}R(R-r)}{r} \iff \\
&\sum_{cyc}\left(\frac{a^2}{s-a}+a\right) \le \frac{6\sqrt{3}R(R-r)}{r} + 2s \iff \\
&\sum_{cyc}\left(\frac{sa}{s-a}\right) \le \frac{6\sqrt{3}R(R-r)}{r} + 2s \iff \\
&s\cdot\left(\frac{4R-2r}{r}\right) \le \frac{6\sqrt{3}R(R-r)}{r} + 2s \iff \\
&s(2R-r) \le 3\sqrt{3}R(R-r) + sr \iff \\
&2s(R-r) \le 3\sqrt{3}R(R-r) \iff \\
&s \le \frac{3\sqrt{3}R}{2}
\end{aligned}$$

and the proof is complete.

4381. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with circumcircle $\mathit{\Gamma}_1$ and circumcenter O. Suppose the open ray AO intersects $\mathit{\Gamma}_1$ at point D and E is the middle point of BC. The perpendicular bisector of BE intersects BD in P and the perpendicular bisector of EC intersects CD in Q. Finally suppose that circle $\mathit{\Gamma}_2$ with center P and radius PE intersects the circle $\mathit{\Gamma}_3$ with center Q and radius QE in X. Prove that AX is a symmedian in $\triangle\, ABC$.

We received 6 submissions, all of which were correct, and feature the solution by K.V. Sudharshan.

Define X' to be the intersection of the A-symmedian with Γ_1. We shall prove that $X' = X$. Let AE intersect Γ_1 at a point Y. Observe that since AX', AY are isogonal (by definition), we have $\angle BAX' = \angle YAC$, and so $BX' = CY$. Since E is the midpoint of BC, we see that $BX'E$ and CYE are congruent triangles (by side-angle-side). Consequently,

$$\angle EX'B = \angle CYE = \angle CYA = \angle CBA.$$

Also, since AD is a diameter of Γ_1 and $P \in BD$, we have $PB \perp AB$, so that $\angle CBA = 90^\circ - \angle PBE$. Thus (because P is on the perpendicular bisector of the segment BE),

$$\angle EPB = 180^\circ - 2 \cdot \angle PBE = 2 \cdot \angle CBA = 2 \cdot \angle EX'B.$$

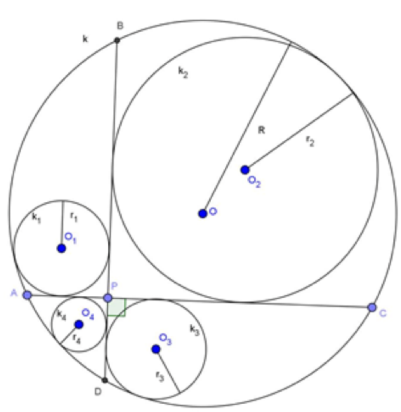
But $\angle EPB$ is the angle at the center of Γ_2 that is subtended by the chord EB, which is twice any angle on the circumference subtended by EB; consequently, X' is a point of Γ_2 as well as lying on Γ_1. Similarly, X' is a point where Γ_3 intersects Γ_1, whence X' is the point other than E where Γ_2 intersects Γ_3. That is, $X' = X$, so that AX must be the A-symmedian, as desired.

Editor's comment. If directed angles are used, then the featured argument does not require the given triangle to be acute: the result holds for an arbitrary ΔABC.

4382. *Proposed by Borislav Mirchev and Leonard Giugiuc.*

Let $ABCD$ be an orthogonal cyclic quadrilateral with $AC \perp BD$. Let O and R be the circumcenter and the circumradius of $ABCD$ respectively and let P be the intersection of AC and BD. Denote by r_1, r_2, r_3 and r_4 the inradii of the minor circular sectors PAB, PBC, PCD and PDA respectively. Prove that

$$r_1 + r_2 + r_3 + r_4 + 8R = (R^2 - OP^2)\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right).$$



We received 7 submissions of which 6 were correct and complete. We feature two of them.

Solution 1, by Sushanth Sathish Kumar.

The proof is by coordinates. Set P to be the origin, and let $O = (x, y)$. Assume the figure is oriented such that B and C lie on the positive y and x-axis, as in the accompanying figure. Let the center of the incircle of sector PBC be O_2.

Note that $O_2 = (r_2, r_2)$, and $OO_2 = R - r_2$. Thus, by the distance formula

$$\begin{aligned}\sqrt{(x-r_2)^2+(y-r_2)^2} = R - r_2 \quad &\iff \quad x^2+y^2-2r_2(x+y)+r_2^2 = R^2-2Rr_2\\ &\iff \quad 2Rr_2-2r_2(x+y)+r_2^2 = R^2-OP^2,\end{aligned}$$

where the last step follows from $OP^2 = x^2 + y^2$. Therefore, we have

$$\frac{R^2-OP^2}{r_2} = 2R + r_2 - 2(x+y).$$

Similarly, we may compute

$$\begin{aligned}\frac{R^2-OP^2}{r_1} &= 2R + r_1 + 2(x-y),\\ \frac{R^2-OP^2}{r_3} &= 2R + r_3 + 2(y-x)\\ \frac{R^2-OP^2}{r_4} &= 2R + r_4 + 2(x+y).\end{aligned}$$

Upon adding, we arrive at

$$(R^2-OP^2)\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right) = r_1+r_2+r_3+r_4+8R,$$

which is precisely the desired result.

Solution 2, by Ioannis D. Sfikas.

Let O_i be the center of the incircle whose radius is r_i, $i = 1, \ldots, 4$. Assuming labelling as in the figure, we have

$$O_1O = R - r_1, \quad O_3O = R - r_3, \quad O_1P = \sqrt{2}r_1, \quad \text{and} \quad O_3P = \sqrt{2}r_3;$$

furthermore, P lies on the line segment O_1O_3. By Stewart's theorem applied to the cevian OP of ΔOO_1O_3 we have

$$OP^2 = (R-r_3)^2\frac{r_1}{r_1+r_3} + (R-r_1)^2\frac{r_3}{r_1+r_3} - 2r_1r_3,$$

so that

$$\frac{R^2-OP^2}{r_1r_3} = 1 + \frac{4R}{r_1+r_3},$$

and, finally,

$$\left(R^2 - OP^2\right)\left(\frac{1}{r_1} + \frac{1}{r_3}\right) = r_1 + r_3 + 4R.$$

Similarly,

$$\left(R^2 - OP^2\right)\left(\frac{1}{r_2} + \frac{1}{r_4}\right) = r_2 + r_4 + 4R.$$

Adding these last two equations gives us the desired result, namely

$$(R^2 - OP^2)\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right) = r_1 + r_2 + r_3 + r_4 + 8R.$$

Editor's comments. It is interesting to compare our problem 4382 with Problem 1.4.6 on pages 9 and 85 of *Japanese Temple Geometry Problems: San Gaku* (edited by Hidetosi Fukagawa and Dan Pedoe and published in 1989 by the Charles Babbage Research Centre). The Japanese problem applies to essentially the same figure, except they use a small circle inside the region bounded by arcs of the given circles and externally tangent to those four given circles (rather than our large circumcircle that encloses the four given circles and is internally tangent to them). This requires replacing our radius R by $-R$. They find a relation among the four radii that involves neither R nor OP, namely

$$r_1r_3(r_2+r_4)^2 + r_2r_4(r_1+r_3)^2 = (r_1r_3-r_2r_4)^2 + (r_1+r_3)(r_2+r_4)(r_1r_3+r_2r_4).$$

4383. *Proposed by Michel Bataille.*

Evaluate the integral

$$I = \int_0^1 (\ln x) \cdot \sqrt{\frac{x}{1-x}}\, dx.$$

We received 14 submissions, all correct. We present the solution by Kee-Wai Lau.

We show that the given integral equals

$$I = \frac{(1 - 2\ln 2)\pi}{2}.$$

By substitution $x = \sin^2\theta$ and using half angle formula $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$, we obtain

$$\begin{aligned} I &= 4\int_0^{\pi/2} \ln(\sin\theta)\sin^2\theta\, d\theta \\ &= 2\left(\int_0^{\pi/2} \ln(\sin\theta)\, d\theta - \int_0^{\pi/2} \ln(\sin\theta)\cos(2\theta)\, d\theta\right). \qquad (1) \end{aligned}$$

The first integral in (1) is well know:

$$\int_0^{\pi/2} \ln(\sin\theta)\,d\theta = \frac{1}{2}\int_0^{\pi} \ln(\sin\theta)\,d\theta = -\frac{\pi\ln 2}{2}$$

(see, for example, p.246 of G. Boros and V. Moll *"Irresistible Integrals"*, Cambridge University Press, 2004.)

Since $\lim_{x\to 0^+} x\ln x = 0$, using a substitution for the second integral of (1), we get

$$\int_0^{\pi/2} \ln(\sin\theta)\cos(2\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/2} \ln(\sin\theta)\,d(\sin(2\theta)) = -\int_0^{\pi/2} \cos^2\theta\,d\theta = -\frac{\pi}{4}.$$

Combining the integrals, we get that $I = \dfrac{(1-2\ln 2)\pi}{2}$ as claimed.

4384. *Proposed by Michel Bataille.*

Let n be an integer with $n \geq 2$. Find all real numbers x such that

$$\sum_{0\leq i<j\leq n-1} \left\lfloor x+\frac{i}{n}\right\rfloor \cdot \left\lfloor x+\frac{j}{n}\right\rfloor = 0.$$

We received 4 correct solutions. We present the composite solution.

For any real number x and integers i, j with $0 \leq i < j \leq n-1$,

$$\lfloor x\rfloor \leq \left\lfloor x+\frac{i}{n}\right\rfloor \leq \left\lfloor x+\frac{j}{n}\right\rfloor < \lfloor x+1\rfloor = \lfloor x\rfloor + 1.$$

Therefore, each term of the sum assumes exactly one of the values a^2, $a(a+1)$ and $(a+1)^2$ for $a = \lfloor x\rfloor$, and so is non-negative. Therefore the sum vanishes if and only if each term $\lfloor x+i/n\rfloor \cdot \lfloor x+j/n\rfloor$ vanishes.

Suppose that $-1/n \leq x < 2/n$. Then

$$0 = \lfloor x+1/n\rfloor = \lfloor x+(n-2)/n\rfloor$$

so that each summand has at least one factor equal to 0 and the equation is satisfied. On the other hand, when $x < -1/n$, then

$$\lfloor x\rfloor \left\lfloor x+\frac{1}{n}\right\rfloor = (-1)^2 > 0$$

and when $x \geq 2/n$, then

$$\left\lfloor x+\frac{n-2}{n}\right\rfloor \cdot \left\lfloor x+\frac{n-1}{n}\right\rfloor = 1 > 0.$$

Therefore the equation is satisfied if and only if $-1/n \leq x < 2/n$.

4385. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let ABC be a triangle with circumcircle ω and $AB < AC$. The tangent at A to ω intersects the line BC at P. The internal bisector of $\angle APB$ intersects the sides AB and AC at E and F, respectively. Show that

$$\frac{PE}{PF} = \sqrt{\frac{EB}{FC}}.$$

We received 14 correct submissions. We present the solution by K. V. Sudharshan.

Let AD be the internal angle bisector of $\angle BAC$, with $D \in BC$. We can see that

$$\angle PAD = \angle PAB + \angle BAD = \angle ACB + \angle CAD = \angle PDA.$$

Thus $PA = PD$. This implies that EF is the perpendicular bisector of AD.

Since AD bisects $\angle EAF$, we see that $EAFD$ is a rhombus, and so $AE = AF$.

Now note that $\triangle PEA \sim \triangle PFC$, so

$$\frac{PE}{PF} = \frac{AE}{FC}.$$

Also, $\triangle PEB \sim \triangle PFA$ and so

$$\frac{PE}{PF} = \frac{BE}{AF}.$$

Combining with $AE = AF$ and the previous result, we have $AE^2 = BE \cdot CF$. Thus,

$$\frac{PE}{PF} = \frac{AE}{FC} = \sqrt{\frac{BE}{CF}}.$$

4386. *Proposed by Thanos Kalogerakis.*

Let $ABCD$ be a cyclic quadrilateral with $AD > BC$, where $X = AB \cap CD$ and $Y = BC \cap AD$. The bisectors of angles X and Y intersect BC and CD at P and S, respectively. Finally, let Q and T be points on the sides AD and AB such that $PQ \perp AD$ and $ST \perp AB$. Prove that $ABCD$ is bicentric if and only if $PQ = ST$.

We received four submissions, but one was withdrawn. We feature the proposer's solution, modified by the editor. The proposer clearly intended for the given quadrilateral to have no parallel sides (rather than demanding inequalities among the edges).

Because a cyclic quadrilateral $ABCD$ has an inscribed circle if and only if the sums of opposite sides are equal, we are to prove that $AD + BC = AB + CD$ if and only if $PQ = ST$. We first show that the area of $ABCD$ is $\frac{1}{2}PQ(AD + BC)$.

From $ABCD$ cyclic we have

$$\angle BAD = \angle BCX. \qquad (1)$$

Denoting by $B'C'$ the reflection of the segment BC in the mirror XP, the triangles PBC' and $PB'C$ are congruent (since P is fixed by the reflection) and, therefore, have the same area. It follows that the quadrangles $ABCD$ and $AC'B'D$ have the same area. Furthermore we have

$$\angle BCX = \angle B'C'X \quad \text{and} \quad BC = B'C'. \tag{2}$$

From (1) and (2) we have $\angle BAD = \angle BCX = \angle B'C'X$, so that the lines AD and $B'C'$ are parallel and, therefore, $AC'B'D$ is a trapezoid with altitude PQ and area

$$\frac{1}{2}PQ(AD + B'C').$$

Because this is also the area of our given quadrilateral and (by (2)) $BC = B'C'$, the area of $ABCD$ is $\frac{1}{2}PQ(AD + BC)$, as claimed.

Similarly, using the same argument with X, XP, PQ, AD, BC replaced by Y, YS, ST, AB, CD, we deduce that the area of $ABCD$ is also given by $\frac{1}{2}ST(AB+CD)$. That is,

$$PQ(AD + BC) = ST(AB + CD),$$

from which we conclude that $AD + BC = AB + CD$ if and only if $PQ = ST$, as desired.

Editor's Comment. If $ABCD$ has an incircle, its diameter has length $PQ = ST$. This follows from the observations that the bisector of $\angle X$ must pass through the incenter, which implies that the reflection in this line fixes the incircle and therefore takes its tangent BC to another tangent $B'C'$; the length of the common perpendicular PQ is the distance between parallel tangents AD and $B'C'$ which, of course, is the diameter of the incircle.

4387. *Proposed by Nguyen Viet Hung.*

Let

$$a_n = \sum_{k=1}^{n} \sqrt[k]{1 + \frac{k^2}{(k+1)!}}, \quad n = 1, 2, 3, \ldots$$

Determine $\lfloor a_n \rfloor$ and evaluate $\lim\limits_{n\to\infty} \dfrac{a_n}{n}$.

We received 7 submissions, of which 6 were correct and complete. We present the solution by Sushanth Sathish Kumar.

We claim that $\lfloor a_n \rfloor = n$. Clearly, $a_n \geq n$, since each term in the sum is at least 1. By the generalized Bernoulli inequality, $(1+x)^{1/k} \leq 1 + \frac{x}{k}$ for any $k \in \mathbb{N}$ and

$x \geq -1$, so in particular we have

$$\begin{aligned} a_n = \sum_{k=1}^{n} \left(1 + \frac{k^2}{(k+1)!}\right)^{\frac{1}{k}} &\leq \sum_{k=1}^{n} \left(1 + \frac{k}{(k+1)!}\right) \\ &= \sum_{k=1}^{n} 1 + \sum_{k=1}^{n} \left(\frac{1}{k!} - \frac{1}{(k+1)!}\right) \\ &= n + \left(1 - \frac{1}{(n+1)!}\right). \end{aligned}$$

Thus, $n \leq a_n < n+1$, which implies $\lfloor a_n \rfloor = n$. By the squeeze theorem,

$$1 = \lim_{n\to\infty} \frac{n}{n} \leq \lim_{n\to\infty} \frac{a_n}{n} \leq \lim_{n\to\infty} \frac{n+1}{n} = 1;$$

that is, $\lim\limits_{n\to\infty} \frac{a_n}{n} = 1$.

4388. *Proposed by Marian Cucoanes and Leonard Giugiuc.*

For positive real numbers a, b and c, prove

$$8abc(a^2 + 2ac + bc)(b^2 + 2ab + ac)(c^2 + 2bc + ab) \leq [(a+b)(b+c)(c+a)]^3.$$

We received 6 submissions, including the one from the proposers. One of the given solutions used Maple outputs. We present the proof by Daniel Văcaru.

The given inequality is equivalent to

$$\frac{(a^2 + 2ac + bc)(b^2 + 2ba + ca)(c^2 + 2cb + ab)}{(a+b)^2(b+c)^2(c+a)^2} \leq \frac{(a+b)(b+c)(c+a)}{8abc} \qquad (1)$$

Let E denote the LHS of (1). Then by direct computations we have

$$E = \Big(\frac{a}{a+b} + \frac{c}{c+a}\Big)\Big(\frac{b}{b+c} + \frac{a}{a+b}\Big)\Big(\frac{c}{c+a} + \frac{b}{b+c}\Big). \qquad (2)$$

Since $a + b \geq 2\sqrt{ab}$, $b + c \geq 2 + \sqrt{bc}$, $c + a \geq 2\sqrt{ca}$, we get from (2) that

$$\begin{aligned} E &\leq \frac{1}{8}\Big(\sqrt{\frac{a}{b}} + \sqrt{\frac{c}{a}}\Big)\Big(\sqrt{\frac{b}{c}} + \sqrt{\frac{a}{b}}\Big)\Big(\frac{c}{c+a} + \frac{b}{b+c}\Big) \\ &= \frac{1}{8abc}(a + \sqrt{bc})(b + \sqrt{ca})(c + \sqrt{ab}) \\ &= \frac{1}{8abc}\big(2abc + ab\sqrt{ab} + bc\sqrt{bc} + ca\sqrt{ca} + a^2\sqrt{bc} + b^2\sqrt{ca} + c^2\sqrt{ab}\big). \qquad (3) \end{aligned}$$

Using AM-GM Inequality again we then obtain from (3) that

$$\begin{aligned} E &\leq \frac{1}{8abc}\Big(2abc + ab\Big(\frac{a+b}{2}\Big) + bc\Big(\frac{b+c}{2}\Big) + ca\Big(\frac{c+a}{2}\Big) \\ &\qquad +a^2\Big(\frac{b+c}{2}\Big) + b^2\Big(\frac{c+a}{2}\Big) + c^2\Big(\frac{a+b}{2}\Big)\Big) \\ &= \frac{1}{8abc}\Big(2abc + a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2\Big) \\ &= \frac{1}{8abc}(a+b)(b+c)(c+a), \end{aligned}$$

so (1) holds and the proof is complete.

4389. *Proposed by Daniel Sitaru.*

Consider the real numbers a, b, c and d. Prove that

$$a(c+d) - b(c-d) \leq \sqrt{2(a^2+b^2)(c^2+d^2)}.$$

We received 21 solutions, all correct, and will feature the solution by Michel Bataille.

The inequality certainly holds if $a(c+d) - b(c-d) < 0$ and otherwise is equivalent to

$$(ac + ad - bc + bd)^2 \leq 2(a^2+b^2)(c^2+d^2).$$

Now, a simple calculation shows that

$$2(a^2+b^2)(c^2+d^2) - (ac + ad - bc + bd)^2 = (ac + bd - ad + bc)^2 \geq 0$$

so we are done.

4390. *Proposed by Marius Drăgan and Neculai Stanciu.*

Let x, y and z be positive real numbers with $x + y + z = m$. Find the minimum value of the expression

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2}.$$

We received 5 submissions of which 2 were correct and complete. We present the solution by the proposers, with minor edits.

We use the following result.

Let $s > 0$ and let $F(x_1, x_2, \ldots, x_n)$ be a symmetrical continuous function on the compact set in $\mathbb{R}^n$

$$S = \{(x_1, x_2, \ldots, x_n)\colon x_1 + x_2 + \cdots + x_n = s, x_1 \geq 0, \ldots, x_n \geq 0\}.$$

If

$$F(x_1, x_2, \ldots, x_n) \geq \min\Big\{F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \ldots, x_n\right), \\ F(0, x_1+x_2, x_3, \ldots, x_n)\Big\} \qquad (1)$$

for all $(x_1, \ldots, x_n) \in S$ with $x_1 > x_2 > 0$, then

$$F(x_1, \ldots, x_n) \geq \min_{1\leq k\leq n} F\left(\frac{s}{k}, \ldots, \frac{s}{k}, 0, \ldots, 0\right),$$

for all $(x_1, \ldots, x_n) \in S$.

A proof of this can be found in *Algebraic inequalities, Old and New Methods*, V. Cârtoaje, Gil Publishing house, 2006, 267-269.

Let $F(x, y, z) = \frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2}$, and let x, y, z be positive reals such that $x + y + z = m$ and $x > y > 0$.

We verify (1) by proving that

$$\text{if } F(x, y, z) < F\left(\tfrac{x+y}{2}, \tfrac{x+y}{2}, z\right), \text{ then } F(x, y, z) \geq F(0, x+y, z).$$

Put $t = \frac{x+y}{2}$, and $p = xy$. Rearranging the inequality $F(x, y, z) < F\left(\frac{x+y}{2}, \frac{x+y}{2}, z\right)$, gives

$$\frac{(t^2-p)(4t^2+2p-2)}{(1+x^2)(1+y^2)(1+t^2)} < 0,$$

and since $t^2 - p > 0$, we have $4t^2 < 2-2p$. It follows that $4t^2p < 2p-2p^2 \leq 2-2p$, and so $2-4t^2p-2p > 0$. The inequality $F(x, y, z) \geq F(0, x+y, z)$ can be rearranged to obtain the equivalent inequality

$$\frac{xy(2-4t^2p-2p)}{(1+x^2)(1+y^2)(1+t^2)} \geq 0,$$

which follows from the above. Thus we can apply the cited result, which gives

$$\begin{aligned} F(x, y, z) &\geq \min\left\{F(m, 0, 0), F\left(\frac{m}{2}, \frac{m}{2}, 0\right), F\left(\frac{m}{3}, \frac{m}{3}, \frac{m}{3}\right)\right\} \\ &= \min\left\{\frac{2m^2+3}{m^2+1}, \frac{m^2+12}{m^2+4}, \frac{27}{m^2+9}\right\} \\ &= \begin{cases} \dfrac{2m^2+3}{m^2+1}, & m \in (0, \sqrt{2}] \\[2ex] \dfrac{m^2+12}{m^2+4}, & m \in (\sqrt{2}, \sqrt{6}] \\[2ex] \dfrac{27}{m^2+9}, & m \in (\sqrt{6}, \infty). \end{cases} \end{aligned}$$

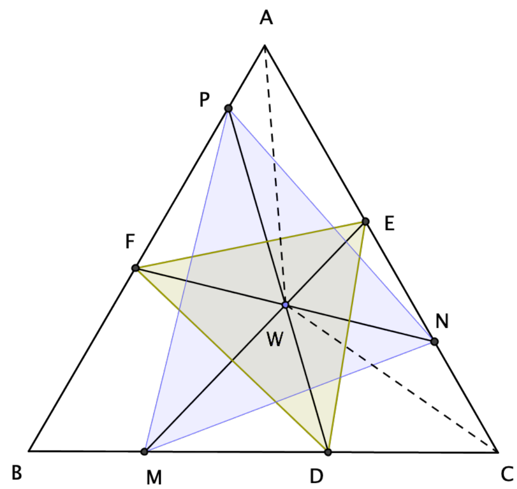
4391. *Proposed by Leonard Giugiuc and Oai Thanh Dao.*

Let ABC be an equilateral triangle and let W be a point inside ABC. A line l_1 through W intersects the segments BC and AB in D and P, respectively. Similarly, a line l_2 through W intersects AC and BC in E and M, and a line l_3 through W intersects AB and AC in F and N. If

$$\angle DWE = \angle EWF = \angle FWD = 120^\circ,$$

show that the triangles DEF and MNP are similar.

We received 6 solutions. We present the solution by C. R. Pranesachar.



It is easy to see that each of the six angles made by l_1, l_2 and l_3 at W is 60°. Thus $MWNC$, $DWEC$ and $EWFA$ each have opposite angles which add up to 180°, and are hence cyclic quadrilaterals. So

$$\angle WNM = \angle WCM = \angle WCD = \angle WED, \text{ and}$$
$$\angle WNP = \angle WAP = \angle WAF = \angle WEF.$$

Adding, we get $\angle MNP = \angle DEF$. Similarly, $\angle NPM = \angle EFD$ and $\angle PMN = \angle FDE$. Thus triangles DEF and MNP are similar.

Editor's Comments. As noted by J. Chris Fisher, this problem is a special consequence of Miquel's theorem. Namely, in any triangle ABC, if we choose points D on BC, E on AC and F on AB, the circumcircles of the three triangles DCE, EAF and FBD intersect at one point, called the Miquel point for DEF. If MNP is constructed analogously and has the same Miquel point as DEF then MNP is similar to DEF. For more details, see Roger A. Johnson's "Advanced Euclidean Geometry", paragraphs 183-188, and problem 1992: 176, 1993: 152-153 proposed by J. Chris Fisher, Dan Pedoe and Robert E. Jamison.

4392. *Proposed by Leonard Giugiuc and Kadir Altintas.*

Let M be an interior point of a triangle ABC with sides $BC = a$, $CA = b$ and $AB = c$. If $MA = x, MB = y$ and $MC = z$, then prove that if

$$\sqrt{(a+y-z)(a-y+z)} + \sqrt{(b+z-x)(b-z+x)} + \sqrt{(c+x-y)(c-x+y)} = \sqrt{3}(x+y+z),$$

then ABC is equilateral.

We received 5 solutions, 4 of which were correct. We present the solution by Sushanth Sathish Kumar.

By Cauchy-Schwarz, we may estimate

$$\begin{aligned} a+b+c &= \sqrt{[(a+y-z)+(b+z-x)+(c+x-y)]} \\ &\quad \cdot \sqrt{[(a-y+z)+(b-z+x)+(c-x+y)]} \\ &\geq \sqrt{(a+y-z)(a-y+z)} + \sqrt{(b+z-x)(b-z+x)} \\ &\quad + \sqrt{(c+x-y)(c-x+y)} \\ &= \sqrt{3}(x+y+z) \end{aligned}$$

But on the other hand, the estimate

$$\sqrt{(a+y-z)(a-y+z)} = \sqrt{a^2-(y-z)^2} \geq a,$$

gives that

$$\sqrt{3}(x+y+z) \geq a+b+c.$$

Thus, it follows that

$$a+b+c = \sqrt{3}(x+y+z).$$

Additionally, we have $x = y = z$, since otherwise equality cannot occur in the second estimate.

Since $x = y = z$, M must be the circumcenter of ABC. Hence, $x = y = z = R$, where R is the circumradius of the triangle. So the equation

$$a+b+c = \sqrt{3}(x+y+z)$$

reduces to

$$\sin A + \sin B + \sin C = \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R} = \frac{3\sqrt{3}}{2}.$$

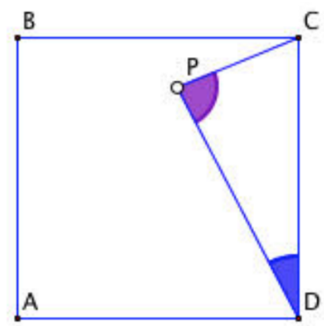
But Jensen's inequality implies that

$$\sin A + \sin B + \sin C \leq 3 \sin\left(\frac{A+B+C}{3}\right) = 3\sqrt{3}/2,$$

with equality holding if and only if $A = B = C = \pi/3$. Consequently, equality must occur in the above equation, which gives $A = B = C = \pi/3$, or that ABC is equilateral.

4393. *Proposed by Ruben Dario Auqui and Leonard Giugiuc.*

Let $ABCD$ be a square. Find the locus of points P inside $ABCD$ such that $\cot \angle CPD + \cot \angle CDP = 2$.



We received 13 solutions, all correct, We feature 4 of them here.

Solution 1, by Paul Bracken and Sushanth Sathish Kumar (done independently).

Assign coordinates $A \sim (0,0)$, $B \sim (0,1)$, $C \sim (1,1)$, $D \sim (1,0)$, $P \sim (x,y)$, and let γ, δ and θ be the respective angles at C, D and P in triangle CDP. Then

$$\cot\gamma = \frac{1-y}{1-x}, \quad \cot\delta = \frac{y}{1-x},$$

and

$$\cot\theta = \frac{1-\cot\gamma\cot\delta}{\cot\gamma+\cot\delta} = \frac{(1-x)^2+(y^2-y)}{1-x}.$$

Therefore

$$2(1-x) = (1-x)(\cot\theta+\cot\delta) = (1-2x+x^2)+y^2,$$

whence $x^2+y^2=1$. The locus of P is a quarter-circle centred at A passing through the vertices B and D.

Solution 2, by Michel Bataille.

Let c, d, p, r be the respective lengths of DP, PC, CD, PA; let h and k be the respective distances from P to CD and AD; let γ, δ, θ be the respective angles at C, D, P in triangle CPD, and let S be the area of triangle CPD.

Noting that

$$2S = pd\sin\gamma = cd\sin\theta = cp\sin\delta = ph,$$

we have that

$$2 = \frac{\cos\theta}{\sin\theta} + \frac{\cos\delta}{\sin\delta} = \frac{\sin(\theta+\delta)}{\sin\theta\sin\delta} = \frac{\sin\gamma}{\sin\theta\sin\delta} = \frac{c^2(pd\sin\gamma)}{(cd\sin\theta)(cp\sin\delta)} = \frac{c^2}{2S},$$

whence $c^2 = 4S = 2ph$.

Then
$$r^2 = k^2 + (p-h)^2 = (c^2-h^2) + (p^2 - 2ph + h^2) = p^2.$$
Therefore, for every position of P, its distance from A is equal to p, so that the locus of P is a quarter-circle with centre A passing through B and D.

Solution 3, by C.R. Pranesachar.

From the Cosine Law, we obtain for any triangle ABC,
$$\cot A = \frac{b^2+c^2-a^2}{2bc\sin A} = \frac{b^2+c^2-a^2}{4[ABC]},$$
where $[ABC]$ is the area of the triangle. Applying this to triangle CPD and using the notation of the previous solution, we have that
$$2 = \cot\theta + \cot\delta = \frac{(c^2+d^2-p^2)+(c^2+p^2-d^2)}{4[CPD]} = \frac{2c^2}{2ph}.$$
Thus $c^2 = 2ph$ and we can complete the argument as in Solution 2.

Solution 4, by Václav Konečný.

Adopt the notation of Solution 2. Let Q and R be the respective feet of the perpendiculars from A and C to the line DP. Then $\angle QAD = \delta$ and so the length of QD is equal to $p\sin\delta$.

The length p of PD is equal to the sum of the lengths of PR and RD when $\theta \le 90^\circ$ and to the difference of these lengths when $\theta > 90^\circ$. The length of RD is equal to $p\cos\delta$. The length of PR is equal to $p\sin\delta\cot\theta$ when $\theta \le 90^\circ$ and $-p\sin\delta\cot\theta$ when $\theta > 90^\circ$. In any case, we find that the length of PD is equal to
$$p\cos\delta + p\sin\delta\cot\theta = p\cos\delta + p\sin\delta(2-\cot\delta) = 2p\sin\delta,$$
i.e. twice the length of QD. It follows that the triangle PAD is isosceles and so the lengths of AP and AD are both equal to p.

Therefore P lies on the quarter-circle with centre A through B and C.

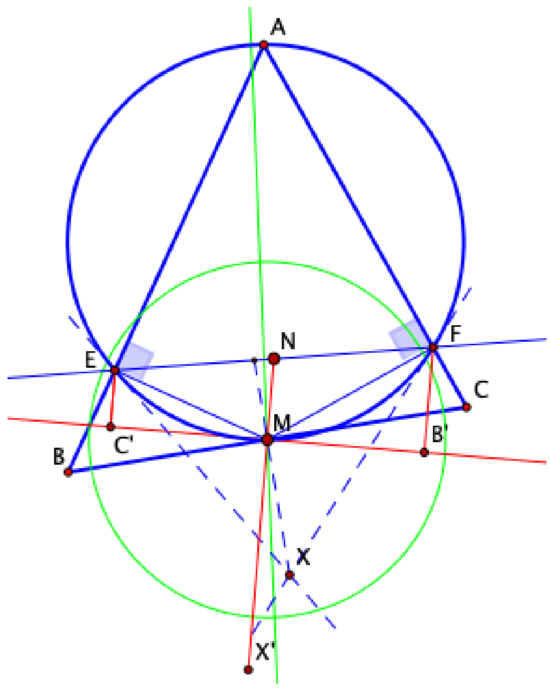
4394. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle and $M \in BC$, $BM \equiv MC$, $E \in AB$, $F \in AC$, $\measuredangle BEM \equiv \measuredangle CFM = 90^\circ$. The two tangents at the points E and F to the circumcircle of $\triangle MEF$ intersect at the point X. If $XM \cap EF = \{Y\}$, show that $YB = YC$.

We received 4 submissions, all correct, and feature two of them.

Solution 1, by Shuborno Das.

To show $YB = YC$, it suffices to show that $YM \perp BC$. But since X, M, and Y are collinear, we just need to prove that $XM \perp BC$. As X is the intersection of the tangents to the circle MEF at E and F, MX is the M-symmedian in ΔMEF. Let's consider a mapping Ψ that is the product of the inversion in the circle centered at M with radius $\sqrt{ME \cdot MF}$ followed by the reflection in the bisector of $\angle EMF$. Note that E and F are interchanged by Ψ. As in the figure, we shall use a prime to denote the image of a point under Ψ.

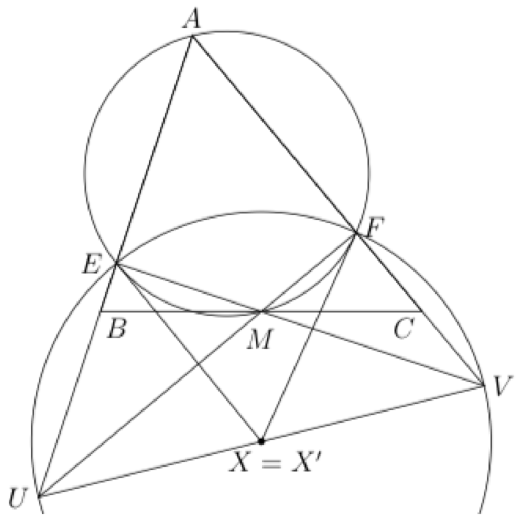


Because $\angle MEB = 90°$, we have $\angle MB'E' = \angle MB'F = 90°$, and similarly, $\angle MC'E = 90°$. Since $MB = MC$ and $M \in BC$, it follows that $MB' = MC'$ and $M \in B'C'$. Moreover, because MX is the M-symmedian in ΔMEF, MX' is the M-median in ΔMEF. Because lines through M are sent by Ψ to their reflection in the bisector of $\angle EMF$, our problem is reduced to showing that $MX' \perp B'C'$. Let MX' meet EF at N; then $NE = NF$. But $MB' = MC'$ and $EC' || FB'$ (because both lines are perpendicular to $B'C'$), while MN is the line that joins the midpoints of $B'C'$ and FE, whence $NM \perp B'C'$, and we are done.

Solution 2, by Sushanth Sathish Kumar.

As in Solution 1, we just need to prove that $XM \perp BC$.

Define $U = FM \cap AE$ and $V = EM \cap AF$. Let X' be the midpoint of UV. We claim that $X = X'$: note that $\angle UEV = \angle BEM = 90° = \angle MFC = \angle UFV$, so X' is the circumcenter of cyclic quadrilateral $UEFV$. As such, triangles $X'UE$, $X'EF$, and $X'FV$ are all isosceles with vertex X'.



Abbreviating $\angle AUV = U$, $\angle AVU = V$ and angle chasing yields

$$\begin{aligned}\angle EX'F &= 180^\circ - \angle UX'E - \angle FX'V \\ &= 180^\circ - (180^\circ - 2U) - (180^\circ - 2V) \\ &= 2U + 2V - 180^\circ.\end{aligned}$$

Thus, from triangle $X'EF$

$$\angle X'EF = \angle X'FE = 180^\circ - U - V = \angle UAV = \angle EAF,$$

which implies that $X'E$ and $X'F$ are tangent to $(AEMF)$. Moreover, this implies X is the circumcenter of $(UEFV)$. Let rays MB, and MC meet $(UEFV)$ at points P, and Q, respectively. By the converse to the butterfly theorem, M is the midpoint of the chord PQ, and (recalling that X is the center of the circle $(UEFV)$) we see this implies that $XM \perp PQ$. Consequently, $XM \perp BC$, as desired.

We end with a projective proof of the converse of the butterfly theorem. Perspectivities from points E and F of the circle $(UPEFUV)$ give

$$(P, M; B, Q) \stackrel{E}{=} (P, V; U, Q) \stackrel{F}{=} (P, C; M, Q),$$

where $(P, M; B, Q)$, is the cross-ratio. Hence (using the hypothesis $BM = MC$),

$$\frac{BP}{BM} \cdot \frac{QM}{QP} = \frac{MP}{MC} \cdot \frac{QC}{QP} \Longrightarrow (BP)(QM) = (MP)(QC).$$

Taking into account that $BP = MP - MB$, we get

$$(MP - MB)(QM) = (MP)(QC) \implies MB(QM) = MP(QM - QC) = MP(MC),$$

which gives $MP = MQ$; thus, M is the midpoint of chord PQ and we are done.

Editor's comments. This editor was unable to find an explicit statement of the butterfly theorem's converse. Many of the vast variety of published proofs of the theorem are reversible, and thus the converse has been tacitly, yet firmly, established. See, for example the first of the editor's proofs in the second volume of ***Crux*** (when the journal was called *Eureka*) [1976: 2-3], or about half of the many proofs presented by Leon Bankoff in "The Metamorphosis of the Butterfly Problem" [*Mathematics Magazine*, **60**:4 (Oct. 1987) 195-210], including a version of the argument above in Solution 2.

4395. *Proposed by Michel Bataille.*

Let $ABCD$ be a tetrahedron and let $A_1, B_1, C_1, A_2, B_2, C_2$ be the midpoints of BC, CA, AB, DA, DB, DC, respectively. Prove that

$$(\overrightarrow{DA} \cdot \overrightarrow{BC})A_1A_2^2 + (\overrightarrow{DB} \cdot \overrightarrow{CA})B_1B_2^2 + (\overrightarrow{DC} \cdot \overrightarrow{AB})C_1C_2^2 = 0,$$

where $\overrightarrow{X} \cdot \overrightarrow{Y}$ denotes the dot product of the vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$.

We received 7 solutions, all of which were correct. We present the solution by Oliver Geupel.

Consider location vectors relative to the origin at point D. We have

$$2\overrightarrow{A_1} = \overrightarrow{B} + \overrightarrow{C}, \;\; 2\overrightarrow{B_1} = \overrightarrow{C} + \overrightarrow{A}, \;\; 2\overrightarrow{C_1} = \overrightarrow{A} + \overrightarrow{B}, \;\; 2\overrightarrow{A_2} = \overrightarrow{A}, \;\; 2\overrightarrow{B_2} = \overrightarrow{B}, \;\; 2\overrightarrow{C_2} = \overrightarrow{C}.$$

We use the alternative notation $\left\langle \overrightarrow{X}, \overrightarrow{Y} \right\rangle$ for the inner product of vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$ for reasons of readability. Let

$$a = \left\langle \overrightarrow{B}, \overrightarrow{C} \right\rangle, \qquad b = \left\langle \overrightarrow{C}, \overrightarrow{A} \right\rangle, \qquad c = \left\langle \overrightarrow{A}, \overrightarrow{B} \right\rangle.$$

It follows that

$$\begin{aligned} 4\left\langle \overrightarrow{DA}, \overrightarrow{BC} \right\rangle A_1A_2^2 &= \left\langle \overrightarrow{A}, \overrightarrow{C} - \overrightarrow{B} \right\rangle \left(\overrightarrow{A} - \overrightarrow{B} - \overrightarrow{C}\right)^2 \\ &= (b-c)\left(\overrightarrow{A}^2 + \overrightarrow{B}^2 + \overrightarrow{C}^2 + 2(a-b-c)\right) \\ &= (b-c)\left(\overrightarrow{A}^2 + \overrightarrow{B}^2 + \overrightarrow{C}^2\right) + 2\left(c^2 - b^2 + ab - ac\right). \end{aligned}$$

With similar identities for the other two terms, we obtain

$$\begin{aligned} &4\left(\left\langle \overrightarrow{DA}, \overrightarrow{BC} \right\rangle A_1A_2^2 + \left\langle \overrightarrow{DB}, \overrightarrow{CA} \right\rangle B_1B_2^2 + \left\langle \overrightarrow{DC}, \overrightarrow{AB} \right\rangle C_1C_2^2\right) \\ &= ((b-c) + (c-a) + (a-b))\left(\overrightarrow{A}^2 + \overrightarrow{B}^2 + \overrightarrow{C}^2\right) \\ &\quad + 2\left(c^2 - b^2 + ab - ca\right) + 2\left(a^2 - c^2 + bc - ab\right) + 2\left(b^2 - c^2 + ca - bc\right) \\ &= 0, \end{aligned}$$

which proves the desired identity.

4396. *Proposed by David Lowry-Duda.*

Show that there is a bijection $f : \mathbb{N} \mapsto \mathbb{N}$ such that the series $\sum_{n=1}^{\infty} \dfrac{1}{n + f(n)}$ converges or show that no such bijection exists.

We received 7 submissions, of which 6 were correct and complete. We present the solution by Oliver Geupel.

Such a bijection exists. For $n \in \mathbb{N}$, let $a_n = n^2$ be the sequence of all perfect squares in ascending order and let

$$b_1 = 2, b_2 = 3, b_3 = 5, b_4 = 6, b_5 = 7, b_6 = 8, b_7 = 10, \dots$$

be the sequence of all positive non-squares in ascending order. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(a_n) = b_n$ and $f(b_n) = a_n$ for all $n \in \mathbb{N}$. This defines a bijection because $\mathbb{N}$ is the disjoint union of $\{a_n : n \in \mathbb{N}\}$ and $\{b_n : n \in \mathbb{N}\}$. For every $k \in \mathbb{N}$ each pair $\{a_k, b_k\}$ appears as $\{n, f(n)\}$ for exactly two distinct values of $n \in \mathbb{N}$. We get

$$\sum_{n=1}^{\infty} \frac{1}{n + f(n)} = 2\sum_{n=1}^{\infty} \frac{1}{a_n + b_n} < 2\sum_{n=1}^{\infty} \frac{1}{a_n} = 2\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3}.$$

Consequently, the series is convergent.

4397. *Proposed by George Stoica.*

Let $n \in \mathbb{N}$ and $k \in \{0, 1, \dots, 2^n\}$. Show that there exists $k' \in \{0, 1, \dots, 2^{n+1}\}$ such that

$$\left| \sin \frac{k'\pi}{2^{n+2}} - \frac{k}{2^n} \right| \le \frac{1}{2^n}.$$

We received 2 solutions, both of which were correct. We present the solution by Omran Kouba.

Let

$$a_m = \sin\left(\frac{\pi m}{2^{n+2}}\right) \text{ for } m \in \{0, 1, \dots, 2^{n+1}\}.$$

Clearly,

$$0 = a_0 < a_1 < \cdots < a_m < a_{m+1} < \cdots < a_{2^n} = 1.$$

Further, for $0 \le x \le y \le \pi/2$, we have

$$0 \le \sin y - \sin x = \int_x^y \cos t \, dt \le \int_x^y dt \le y - x.$$

So, since $\pi < 4$, we have

$$a_{m+1} - a_m < \frac{\pi}{2^{n+2}} < \frac{1}{2^n}, \quad \text{for } m = 0, 1, \dots, 2^{n+1} - 1.$$

Now, given $k \in \{0, 1, \dots, 2^n\}$ we consider

$$\mathcal{N}_k = \left\{ m \in \{0, 1, \dots, 2^{n+1}\} : a_m \le \frac{k}{2^n} \right\}.$$

Clearly, $\mathcal{N}_k \neq \emptyset$ because it contains 0 and it has 2^{n+1} as an upper bound. So we may define $k' = \max \mathcal{N}_k$.

- If $k' = 2^{n+1}$ then $a_{k'} = 1$; this corresponds to the case $k = 2^n$ and the desired inequality holds trivially in this case.
- If $k' < 2^{m+1}$ then by definition of k' we have $a_{k'} \leq k2^{-n} < a_{k'+1}$ so

$$0 < \frac{k}{2^n} - a_{k'} < a_{k'+1} - a_{k'} < \frac{1}{2^n}.$$

and the desired inequality follows.

4398. *Proposed by Daniel Sitaru.*

Prove that for $n \in \mathbb{N}^*$, we have

$$\frac{1}{2n-1} + \int_0^1 \sin^2(x^n)dx \geq \frac{2}{n}(1 - \cos 1).$$

We received 10 submissions, including the one from the proposer, all of which are correct. We present the nearly identical solution by Michel Bataille, Leonard Giugiuc, Digby Smith, and Daniel Văcaru.

Since $a^2 + b^2 \geq 2ab$ for all $a, b \in \mathbb{R}$, we have

$$\begin{aligned}
\frac{1}{2n-1} + \int_0^1 \sin^2(x^n)dx &= \int_0^1 (x^{2n-2} + sin^2(x^n))dx \\
&\geq 2\int_0^1 x^{n-1} \sin(x^n)dx \\
&= -\frac{2}{n}\cos(x^n)\Big|_0^1 \\
&= -\frac{2}{n}(\cos 1 - \cos 0) \\
&= \frac{2}{n}(1 - \cos 1).
\end{aligned}$$

4399. *Proposed by Lacin Can Atis.*

Let $ABCDE$ be a pentagon. Prove that

$$|AB||EC||ED| + |BC||EA||ED| + |CD||EA||EB| \geq |AD||EB||EC|.$$

When does equality hold?

We received 7 solutions. We present the one by Michel Bataille.

We shall use the notation XY instead of $|XY|$. We observe that the left-hand side $\mathcal{L}$ of the inequality rewrites as

$$\begin{aligned}\mathcal{L} &= ED\cdot(AB\cdot EC+EA\cdot BC-EB\cdot AC)\\ &\quad +EB\cdot(ED\cdot AC+EA\cdot CD-AD\cdot EC)+AD\cdot EB\cdot EC.\end{aligned}$$

From Ptolemy's inequality, we have

$$AB\cdot EC+EA\cdot BC-EB\cdot AC\geq 0$$

with equality if and only if A, B, C, E lie, in this order, on a circle and

$$ED\cdot AC+EA\cdot CD-AD\cdot EC\geq 0$$

with equality if and only if A, C, D, E lie, in this order, on a circle.

It follows that $\mathcal{L}\geq AD\cdot EB\cdot EC$, the desired inequality, and that equality holds if and only if $ABCDE$ is a cyclic pentagon.

4400. *Proposed by Daniel Sitaru.*

Prove that in any triangle ABC, the following relationship holds:

$$\sum_{cyc}\frac{\sin(\frac{\pi}{3}-\frac{A}{2})\sin(\frac{\pi}{3}-\frac{B}{2})}{\cos(\frac{C-A}{2})\cos(\frac{C-B}{2})}<1.$$

We received 6 submissions, including the one from the proposer. As it turned out, the proposed inequality is false, and four of the five solvers give various counterexamples. We will feature below some of the given solutions and comments.

Counterexample 1, given by Leonard Giugiuc, enhanced by the editor.

Let E denote the LHS of the given inequality. Set $A=B\in(0,\frac{\pi}{2})$ so $C=\pi-2A$. Then

$$\frac{\sin(\frac{\pi}{3}-\frac{A}{2})\sin(\frac{\pi}{3}-\frac{B}{2})}{\cos(\frac{C-A}{2})\cos(\frac{C-B}{2})}=\frac{\sin^2(\frac{\pi}{3}-\frac{A}{2})}{\cos^2(\frac{\pi-3A}{2})},\tag{1}$$

$$\begin{aligned}\frac{\sin(\frac{\pi}{3}-\frac{B}{2})\sin(\frac{\pi}{3}-\frac{C}{2})}{\cos(\frac{A-B}{2})\cos(\frac{A-C}{2})}&=\frac{\sin(\frac{\pi}{3}-\frac{A}{2})\sin(A-\frac{\pi}{6})}{\cos(\frac{3A-\pi}{2})}\\ &=\frac{\sin(\frac{\pi}{3}-\frac{A}{2})\sin(A-\frac{\pi}{6})\cos(\frac{\pi-3A}{2})}{\cos^2(\frac{\pi-3A}{2})},\end{aligned}\tag{2}$$

and

$$\begin{aligned}\frac{\sin(\frac{\pi}{3}-\frac{C}{2})\sin(\frac{\pi}{3}-\frac{A}{2})}{\cos(\frac{B-C}{2})\cos(\frac{B-A}{2})}&=\frac{\sin(A-\frac{\pi}{6})\sin(\frac{\pi}{3}-\frac{A}{2})}{\cos(\frac{3A-\pi}{2})}\\ &=\frac{\sin(A-\frac{\pi}{6})\sin(\frac{\pi}{3}-\frac{A}{2})\cos(\frac{3A-\pi}{2})}{\cos^2(\frac{\pi-3A}{2})}.\end{aligned}\tag{3}$$

From (1)+(2)+(3) we then obtain

$$E = \frac{1}{\cos^2(\frac{\pi - 3A}{2})}\left(\sin^2\left(\frac{\pi}{3} - \frac{A}{2}\right) + 2\sin\left(\frac{\pi}{3} - \frac{A}{2}\right)\sin\left(A - \frac{\pi}{6}\right)\cos\left(\frac{3A - \pi}{2}\right)\right).$$

Now, let $A \to 0^+$. Then

$$\cos^2\left(\frac{\pi - 3A}{2}\right) \to 0^+, \sin\left(\frac{\pi}{3} - \frac{A}{2}\right) \to \frac{\sqrt{3}}{2}, \text{ and } \sin\left(A - \frac{\pi}{6}\right) \to -\frac{1}{2}$$

so

$$\lim_{A \to 0^+} E = \infty.$$

Counterexample 2, by Alexandru Daniel Pîrvuceanu, with all the details supplied by the editor.

Let $A = 150°, B = C = 15°$. Then with calculations carried to 4 decimal places, we have:

$$\frac{\sin(\frac{\pi}{3} - \frac{A}{2})\sin(\frac{\pi}{3} - \frac{B}{2})}{\cos(\frac{C-A}{2})\cos(\frac{C-B}{2})} = \frac{-(\sin 15°)(\sin 52.5°)}{\cos(67.5°)\cos 0°} = -0.5366$$

$$\frac{\sin(\frac{\pi}{3} - \frac{B}{2})\sin(\frac{\pi}{3} - \frac{C}{2})}{\cos(\frac{A-B}{2})\cos(\frac{A-C}{2})} = \frac{\sin^2(52.5°)}{\cos^2(67.5°)} = 4.2980$$

$$\frac{\sin(\frac{\pi}{3} - \frac{C}{2})\sin(\frac{\pi}{3} - \frac{A}{2})}{\cos(\frac{B-C}{2})\cos(\frac{B-A}{2})} = \frac{\sin(52.5°)(-\sin 15°)}{\cos(67.5°)} = -0.5366$$

Hence, $E = 4.2980 - 2(0.5366) = 3.2248 > 1$.

Editor's note. Digby Smith remarked that the given inequality holds if each of $A, B, C > \frac{\pi}{6}$, and Pranesachar gave examples to show that $E < 1, E = 1$, and $E > 1$ are all possible.