OC440. Soit $f : [a, b] \to [a, b]$ une fonction différentiable dont la dérivée première est continue et positive. Prouvez qu'il existe $c \in (a, b)$ tel que

$$f(f(b)) - f(f(a)) = (f'(c))^2(b - a).$$

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(8), p. 324–325; 44(9): 370-371; 44(10): 412-413.

OC396. Prove that there are infinitely many positive integers m such that the number of odd distinct prime factors of $m(m + 3)$ is a multiple of 3.

Originally Problem 5 from the Final Round of 2017 Italy Math Olympiad.

We received no submissions for this problem.

OC397. In a triangle ABC with $\angle A = 45°$, draw the median AM. The line b is symmetrical to the line AM with respect to the altitude BB_1 and the line c is symmetrical to AM with respect to the altitude CC_1. The lines b and c intersect at the point X. Prove that $AX = BC$.

Originally Problem 6 from Grade 9 competition of the 2017 Moscow Math Olympiad.

We received 3 correct submissions. We present two solutions.

Solution 1, by Oliver Geupel.

Put the triangle onto a complex plane such that the unit circle (O) is circumscribed about triangle ABC and identify each point with the corresponding complex number. By the hypothesis $\angle A = 45°$, there is no loss of generality in putting $B = -(1 + i)/\sqrt{2}$ and $C = (1 - i)/\sqrt{2}$. Then,

$$M = \frac{B + C}{2} = \frac{-i}{\sqrt{2}}, \qquad \bar{M} = \frac{i}{\sqrt{2}}.$$

Let (O) intersect the lines AM, BB_1, and CC_1 for the second time at points D, E, and F, respectively. Since A and D are complex numbers of absolute value 1, the equation of the line AD is $Z = A + D - AD\bar{Z}$. We put $Z = M$ and solve for D, obtaining

$$D = \frac{M - A}{1 - A\bar{M}} = \frac{A\sqrt{2} + i}{Ai - \sqrt{2}}.$$

From $\angle EOA = 2\angle EBA = 90^\circ$ and the similar relation $\angle AOF = 90^\circ$, we deduce

$$E = A/i, \qquad F = Ai.$$

Let

$$Y = \frac{-Ai}{A\sqrt{2} + i}.$$

We shall show that $Y = X$ and that $AY = BC$. Indeed,

$$AY^2 = (A - Y)\overline{(A - Y)} = \frac{A^2\sqrt{2} + 2Ai}{A\sqrt{2} + i} \cdot \frac{\sqrt{2}/A^2 - 2i/A}{\sqrt{2}/A - i} = 2 = BC^2.$$

The orthogonal projection of Y onto the chord BE is $P = (B + E + Y - BE\bar{Y})/2$. The mirror image Q of Y under reflection in the axis BE satisfies $Q - P = P - Y$. Hence,

$$Q = 2P - Y = B + E - BE\bar{Y} = \frac{1 - i - A^2 i}{A + i\sqrt{2}}.$$

Similarly, the mirror image of Y under reflection in the axis CF is

$$S = C + F - CF\bar{Y} = \frac{A^2 + 1 - i}{\sqrt{2} - Ai}.$$

Since

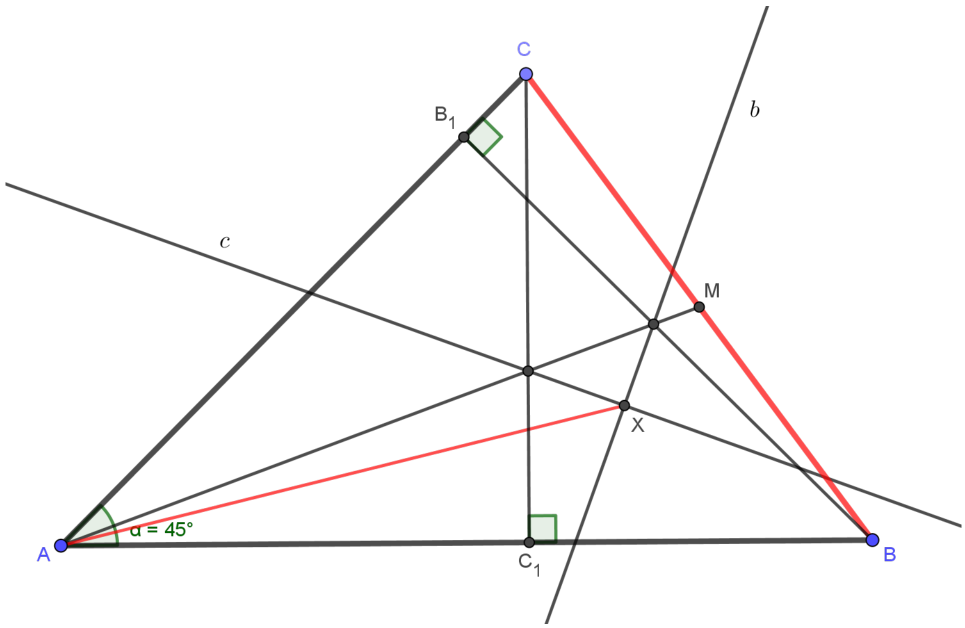
$$AD\bar{Q} = A \cdot \frac{A\sqrt{2} + i}{Ai - \sqrt{2}} \cdot \frac{1 + i + (i/A^2)}{(1/A) - i\sqrt{2}} = \frac{A^2 i - A^2 - 1}{Ai - \sqrt{2}} = A + D - Q,$$

the point Q belongs to the chord AD. Hence, the point Y lies on the line b. Analogously, $AD\bar{S} = A + D - S$, which implies that S belongs to AD, and Y lies on the line c. Consequently, $Y = X$ and $AX = AY = BC$.

Solution 2, by Andrea Fanchini.

We use barycentric coordinates with reference to the triangle ABC. Therefore, we have

$$AM : y - z = 0, \qquad BB_1 : S_A x - S_C z = 0, \qquad CC_1 : S_A x - S_B y = 0.$$



The line b, symmetrical to the line AM with respect to the altitude BB_1, is

$$b : 2S_A x - (S_A + S_C)y + (S_A - S_C)z = 0$$

and the line c, symmetrical to AM with respect to the altitude CC_1, is

$$c : 2S_A x + (S_A - S_B)y - (S_A + S_B)z = 0.$$

The lines b and c intersect at the point X

$$X = (a^2 : 2S_A + S_B - S_C : 2S_A - S_B + S_C),$$

and the distance between the points X and A is

$$\begin{aligned} AX^2 &= \frac{(S_A + S_B)(2S_A + S_B - S_C)^2 + 2S_A(2S_A + S_B - S_C)(2S_A - S_B + S_C)}{(4S_A + a^2)^2} \\ &+ \frac{(S_A + S_C)(2S_A - S_B + S_C)^2}{(4S_A + a^2)^2} \\ &= \frac{4S_A^2 + (S_B - S_C)^2}{4S_A + a^2}. \end{aligned}$$

Since $\angle A = 45^\circ$, then $S_A = S$, and

$$AX^2 = \frac{4a^2 S_A + (S_B + S_C)^2}{4S_A + a^2} = \frac{a^2(4S_A + a^2)}{4S_A + a^2} = a^2.$$

So $AX = BC = a$.

OC398. Detective Nero Wolfe is investigating a crime. There are 80 people involved in this case, among them one is the criminal and another is a witness of the crime (but it is not known who is who). Every day, the detective can invite one or more of these 80 people for an interview; if among the invited there is the witness, but there is no criminal, then the witness will tell who the criminal is. Can the detective solve the case in 12 days?

Originally Problem 3 of Grade 11 competition of the 2017 Moscow Math Olympiad.

We received 3 correct submissions. We present two solutions followed by a generalization of the question.

Solution 1, by Kathleen Lewis.

Yes, the detective can solve the case in 12 days.

Number the 80 people from 1 to 80, and convert all 80 numbers to base 3. All base 3 numbers have 4 or fewer digits. Add leading zeroes to the smaller numbers so that all of them have 4 digits. Then Detective Wolfe's strategy is as follows: the detective invites all people whose ones digit is 0 on day 1, all people whose ones digit is 1 on day 2, and all people whose ones digit is 2 on day 3. He continues using the threes, nines and 27s digits. This process will take him exactly $3 \times 4 = 12$ days. Since any two people must have numbers that differ in at least one digit, there must be at least one group that contains the witness but not the criminal.

Solution 2 and generalization by Oliver Geupel.

Yes, the detective can solve the case, even in 9 days. We propose a strategy that solves, for any natural number n, a case with a number $p \leq \binom{n}{\lfloor n/2 \rfloor}$ of people in n days. Since p is less than or equal to the number of subsets of size $\lfloor n/2 \rfloor$ of a set of size n, we can use these subsets to construct a matrix $A = (a_{ik})$ of size $p \times n$ with the following properties. All entries of A are either 1 or 0. Each row has exactly $\lfloor n/2 \rfloor$ entries that are equal to 1. Any two rows of A are not the same.

The matrix A is used to build our strategy. Identify the p people with the integers $1, 2, \ldots, p$. For each $1 \leq k \leq n$, the people that are invited on the k-th day are those people with numbers i such that $a_{ik} = 1$. By hypothesis, for every pair $i \neq j$ the rows i and j are not the same, and hence, there are column indices k and ℓ such that $a_{ik} = a_{j\ell} = 0$ and $a_{i\ell} = a_{jk} = 1$. Then, among those invited at day k, there is the person j but not the person i, and among those invited at day ℓ, there is the person i but not the person j. Thus, on some day, the witness but not the criminal is invited.

In our case we have 80 people and

$$80 < \binom{9}{\lfloor 9/2 \rfloor} = 126,$$

hence the detective has a strategy to find the criminal in 9 days.

OC399. We say that a function $f : \mathbb{R} \to \mathbb{R}$ has the property $\mathcal{P}$ if for any sequence of real numbers $(x_n)_{n\geq 1}$ such that the sequence $(f(x_n))_{n\geq 1}$ converges, then also the sequence $(x_n)_{n\geq 1}$ converges. Prove that a surjective function with property $\mathcal{P}$ is continuous.

Originally Problem 1 of Grade 11 competition of the 2017 Romania Math Olympiad.

We received 2 correct submissions. We present the solution by the Missouri State University Problem Solving Group.

We start by establishing three facts.

Fact 1. Let $g : \mathbb{R} \to \mathbb{R}$. If g is continuous and injective, then it is strictly monotone.

We prove Fact 1 by contradiction. Assume g is continuous and injective, but not strictly monotone. Without loss of generality, assume there exists $a < b < c$ with $g(a) < g(c) < g(b)$. Let $k \in \mathbb{R}$ satisfy $g(a) < g(c) < k < g(b)$. Then by the Intermediate Value Theorem, there exists $a < x_1 < b < x_2 < c$ with $f(x_1) = f(x_2) = k$. This contradicts the assumption that g is injective. Therefore g must be strictly monotone. $\square$

Fact 2. Let $g : \mathbb{R} \to \mathbb{R}$. If g is monotone and surjective, then it is continuous.

We prove Fact 2 by contradiction. Assume g is monotone and surjective, but discontinuous at a. Without loss of generality, we can futher assume that g is increasing. Since g is monotone, the discontinuity must be a jump discontinuity and

$$\alpha = \sup\{g(x) : x < a\} < \inf\{g(x) : x > a\} = \beta.$$

Since $g : \mathbb{R} \to \mathbb{R}$ is surjective, there at least two distinct real numbers s and t such that $\alpha < g(s) < g(t) < \beta$. However, this contradicts that only $g(a)$ can possibly be between α and β. Therefore g must be continuous. $\square$

Fact 3. Given any two sequences (a_n) and (b_n), define the sequence

$$(a_n \sqcup b_n) := (a_1, b_1, a_2, b_2, a_3, b_3, \dots).$$

The sequence $(a_n \sqcup b_n)$ converges if and only if both sequences (a_n) and (b_n) converge to the same limit.

The proof of Fact 3 is trivial and is not included here.

Using these three facts, we proceed to prove the main question. Assume $f : \mathbb{R} \to \mathbb{R}$ is surjective and has property $\mathcal{P}$.

Let a and b be two real numbers such that $f(a) = f(b)$. Define the constant and convergent sequence

$$(f(a) \sqcup f(b)) = (f(a), f(b), f(a), f(b), \dots).$$

By property $\mathcal{P}$, the sequence $(a \sqcup b)$ also converges. Then $a = b$, and so f is injective. Now f is a bijection with inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$.

Assume the sequence (y_n) converges to y, and let $x_n = f^{-1}(y_n)$ and $x = f^{-1}(y)$. Then $(f(x_n) \sqcup f(x)) = (y_n \sqcup y)$ converges, and by property $\mathcal{P}$, $(x_n \sqcup x)$ converges. Using Fact 3, it follows that

$$\lim_{n\to\infty} f^{-1}(y_n) = \lim_{n\to\infty} x_n = x = f^{-1}(y),$$

and so f^{-1} is continuous. Now using Fact 1 for the continuous and injective function f^{-1} we get that f^{-1} is monotone. Then f must be monotone, as well. Lastly, using Fact 2 for the monotone and surjective function f we obtain that f is continuous, as required.

OC400. Let G be a finite group having the following property: for any automorphism f of G, there exists a natural number m such that $f(x) = x^m$ for all $x \in G$. Prove that G is abelian.

Originally Problem 3 of Grade 12 competition of the 2017 Romania Math Olympiad.

We received 1 correct submission by Oliver Geupel which is presented next.

Let n be the order of G. Let $o(g)$ denote the order of an element g of G. For any prime number p, let H_p be the set of all those elements of G, whose order is a divisor of p^n, that is, $H_p = \{g \in G : o(g) \mid p^n\}$.

Fact 1. Let $g, h \in G$ such that $o(g) \mid o(h)$. Then, g and h commute.

Proof. The conjugation by h, $x \mapsto hxh^{-1}$, is an automorphism of G. By hypothesis, there is a natural number m such that $hxh^{-1} = x^m$ for every $x \in G$. Putting $x = h$, we find that $h = h^m$; whence $o(h) \mid m-1$ and $o(g) \mid m-1$. Putting $x = g$, we obtain $hgh^{-1} = g^m = g$ and thus $hg = gh$. □

Fact 2. H_p constitutes an abelian subgroup of G for every prime number p.

Proof. The identity element e which has order 1, is in H_p. For every element g of H_p, its inverse g^{-1} is of the same order as g; whence $g^{-1} \in H_p$. For any two elements of H_p, the order of one element divides the order of the other one by the definition of H_p. Hence, by Fact 1, the two elements commute. Thus, for $g, h \in H_p$, we have $(gh)^{p^n} = g^{p^n} h^{p^n} = e$, which implies $gh \in H_p$. □

Fact 3. Let p and q be distinct primes and let $g \in H_p$ and $h \in H_q$. Then, g and h commute.

Proof. Since $(hgh^{-1})^{p^n} = e$, it holds $hgh^{-1} \in H_p$. Applying Fact 2, we obtain $g^{-1} \in H_p$ and $hgh^{-1}g^{-1} = (hgh^{-1})g^{-1} \in H_p$. Analogously, we deduce $hgh^{-1}g^{-1} \in H_q$. But $H_p \cap H_q = \{e\}$. It follows that $hgh^{-1}g^{-1} = e$. Consequently, $hg = gh$. □

We are now prepared for the proof that G is abelian. Let $n = \prod_{k=1}^{\ell} p_k^{a_k}$ be the canonical factorisation of n into primes. Then, the numbers $d_k = n/p_k^{a_k}$ are coprime for $k = 1, \ldots, \ell$. Hence, there exist integers $c_1, \ldots, c_\ell$ such that

$\sum_{k=1}^{\ell} c_k d_k = 1$. Let $g, h \in G$. It holds

$$g = \prod_{k=1}^{\ell} (g^{d_k})^{c_k}, \quad h = \prod_{k=1}^{\ell} (h^{d_k})^{c_k},$$

where $g^{d_k}, h^{d_k} \in H_{p_k}$, $k = 1, \ldots, \ell$. From Facts 2 and 3, we finally conclude $gh = hg$. This proves that G is an abelian group.

OC401. Determine all polynomials $P(x) \in \mathbb{R}[x]$ satisfying the following two conditions:

(a) $P(2017) = 2016$;

(b) $(P(x) + 1)^2 = P(x^2 + 1)$ for all real numbers x.

Originally Problem 1 from Final Round of the 2017 Austria Math Olympiad.

We received 6 *submissions. We present the solution by Oliver Geupel.*

A solution is $P(x) = x - 1$, and we prove that it is unique. Suppose $P(x)$ is a solution. Define a sequence $(x_n)_{n \in \mathbb{N}}$ by the recursion

$$x_1 = 2017, \qquad x_{n+1} = x_n^2 + 1 \quad (n \in \mathbb{N}).$$

We show by mathematical induction that $P(x_n) = x_n - 1$ holds for every positive integer n. The base case $n = 1$ is settled by condition (a). Assuming that for some specific n we have $P(x_n) = x_n - 1$, it follows by condition (b) that

$$P(x_{n+1}) = P(x_n^2 + 1) = (P(x_n) + 1)^2 = x_n^2 = x_{n+1} - 1,$$

which completes the induction. As a consequence, the infinitely many numbers $x_1, x_2, x_3, \ldots$ are roots of the polynomial $Q(x) = P(x) - x + 1$. Hence, Q is the null polynomial and $P(x) = x - 1$.

Editor's Comments. Walther Janous, who is also the author of this problem, investigated the case in which only condition (b) holds. We present his analysis.

With the substitution $Q(x) := P(x) + 1$, the functional equation given in (b) becomes

$$Q(x^2 + 1) = ((Q(x))^2 + 1, \qquad x \in \mathbb{R},$$

which we call (FG). From now on, we will look exclusively at (FG). Assuming that $Q(x) = a + bx + cx^2 + \cdots$, by a coefficient comparison, we get for small degrees of Q the following polynomials:

$$\begin{aligned} Q_0(x) &= x && \text{if } \deg Q = 1 \\ Q_1(x) &= x^2 + 1 && \text{if } \deg Q = 2 \\ Q_2(x) &= x^4 + 2x^2 + 2 && \text{if } \deg Q = 4 \\ Q_3(x) &= x^8 + 4x^6 + 8x^4 + 8x^2 + 5 && \text{if } \deg Q = 8 \end{aligned}$$

as solutions of the functional equation (FG).

There are no solutions Q with $\deg Q \in \{0, 3, 5, 6, 7\}$. In addition one recognizes that these polynomials satisfy the recurrence relation $Q_{n+1}(x) = (Q_n(x))^2 + 1$, $x \in \mathbb{R}$ for $n \in \{0, 1, 2\}$. Conversely, every polynomial $Q_n(x)$ satisfying this recurrence relation with $Q_0(x) = x$ is a solution of (FG). This is evident for $Q_0(x)$. Let $Q_n(x)$ be a solution of (FG). Then, in particular $Q_n(x^2 + 1) = (Q_n(x))^2 + 1$, so also $(Q_n(x^2 + 1))^2 + 1 = ((Q_n(x))^2 + 1)^2 + 1$, i.e. $Q_{n+1}(x^2 + 1) = (Q_{n+1}(x))^2 + 1$. Thus, $Q_{n+1}(x)$ is also a solution of (FG). For the general solution of the functional equation (FG) we need the following three lemmas.

Lemma. Let $P(x) \in \mathbb{R}[x]$ be a polynomial with $P(0) = 0$ and let f be a real-valued function with $f(x) > x$ for every $x \in \mathbb{R}$. Then, the functional equation $P(f(x)) = f(P(x))$, $x \in \mathbb{R}$, has the polynomial $P(x) = x$, $x \in \mathbb{R}$, as the only solution.

Proof. We define the sequence $(x_n)_{n\geq 0}$ recursively by $x_0 = 0$ and $x_{n+1} = f(x_n)$, $n \geq 0$. We prove by induction that $P(x_n) = x_n$, $n \geq 0$. By hypothesis, we have $P(0) = 0$, i.e $P(x_0) = x_0$. Let $P(x_k) = x_k$, where $k \geq 0$. Then, $P(x_{k+1}) = P(f(x_k)) = f(P(x_k)) = f(x_k) = x_{k+1}$. Moreover, $x_{k+1} = f(x_k) > x_k$, $k \geq 0$. This gives us a sequence of points $x_0 < x_1 < x_2 < \ldots$ for which the polynomials $P(x)$ and $\mathrm{id}(x) = x$ coincide at all points of the sequence. So, it must be $P(x) = x$, $x \in \mathbb{R}$, and the conclusion follows. □

Lemma. All polynomials Q that satisfy the functional equation (FG) are either even or odd.

Proof. Since $(Q(-x))^2 = Q((-x)^2 + 1) - 1 = Q(x^2 + 1) - 1 = ((Q(x))^2$, $x \in \mathbb{R}$, then for every $x \in \mathbb{R}$ we have

$$Q(-x) = Q(x) \qquad \text{or} \qquad Q(-x) = -Q(x).$$

That is, at least one of these two relations is fulfilled for an infinite number of $x \in \mathbb{R}$. Since Q is a polynomial, therefore, either $Q(-x) = Q(x)$, $x \in \mathbb{R}$ or $Q(-x) = -Q(x)$, $x \in \mathbb{R}$. So, the polynomial Q is either even or odd. □

Lemma. If a polynomial Q with $Q(0) \neq 0$ is the solution of the functional equation (FG), then there exists a polynomial S, with $\deg S = \dfrac{1}{2} \deg Q$, which also satisfies (FG), where $Q(x) = S(x^2 + 1)$, $x \in \mathbb{R}$.

Proof. The second lemma and $Q(0) \neq 0$ show that Q must be even, i.e. $Q(x) = R(x^2)$, $x \in \mathbb{R}$, with $R \in \mathbb{R}[x]$. Therefore, the functional equation (FG) can be written in the form $R((x^2 + 1)^2) = (R(x^2))^2 + 1$, $x \in \mathbb{R}$. The variable substitution $\xi := x^2 + 1$ gives

$$R(\xi^2) = (R(\xi - 1))^2 + 1 \implies R((\xi^2 + 1) - 1) = (R(\xi - 1))^2 + 1$$

for all $\xi \in [1, \infty)$. Since R is a polynomial, this relation holds even for every $\xi \in \mathbb{R}$. Therefore, with the function substitution $S(z) := R(z - 1)$, $z \in \mathbb{R}$, we get $S(\xi^2 + 1) = (S(\xi))^2 + 1$ for $\xi \in \mathbb{R}$, so S is also a solution of (FG). □

Let us go back to the solution of the functional equation (FG). We show by induction that for $n \geq 0$ there exists exactly one polynomial Q, with $2^n \leq \deg Q < 2^{n+1}$, which is a solution of (FG), namely Q_n. For $n = 0$, that is $\deg Q = 1$, the assertion is proved by setting $Q(x) = ax + b$ and by comparison of coefficients.

We assume that the statement is true for $n \geq 0$ and show that it then also holds for $n+1$. Let Q be a polynomial with $2^{n+1} \leq \deg Q < 2^{n+2}$ which is a solution of (FG). We have two cases.

(i) $Q(0) = 0$. Then, the first lemma applied to the function $f(x) = x^2 + 1$, $x \in \mathbb{R}$ that satisfies $f(x) > x$, $x \in \mathbb{R}$, implies that $Q(x) = x$, $x \in \mathbb{R}$. But this is not possible because $\deg Q \geq 2$.

(ii) $Q(0) \neq 0$. Then, by the second lemma Q must be even. By the third lemma, we have $Q(x) = S(x^2 + 1)$, where S satisfies the functional equation (FG), $\deg S = \dfrac{1}{2} \deg Q$ and $2^n \leq \deg S < 2^{n+1}$. By the induction hypothesis, $S = Q_n$ and thus $Q = Q_{n+1}$.

The general solutions to the functional equation considered in part (b) of the problem statement are therefore $P_n(x) = Q_n(x) - 1$. If $Q_0(2017) = 2017$, we obtain $Q_{n+1}(x) = (Q_n(x))^2 + 1 > (Q_n(x))^2$, $x \in \mathbb{R}$, $n \geq 0$, which gives immediately $Q_n(2017) > 2017$ for all $n \geq 1$. Therefore, $P(x) = x - 1$, $x \in \mathbb{R}$, is the only polynomial that satisfies the two conditions of the problem.

OC402. Find all natural numbers n that satisfy the following property: for each integer $k \geq n$ there is a multiple of n whose digits sum up to k.

Originally Problem 5 from Grade 10 competition of the 2017 Moscow Math Olympiad.

We received no submissions to this problem.

OC403. Let S be the point of tangency of the incircle of a triangle ABC with the side AC. Let Q be a point such that the midpoints of the segments AQ and QC lie on the incircle. Prove that QS is the angle bisector of $\angle AQC$.

Originally Problem 2 from Grade 11 competition of the 2017 Moscow Math Olympiad.

We received no submissions to this problem.

OC404. Let $(A, +, \cdot)$ be a ring simultaneously satisfying the conditions:

(i) A is not a division ring;

(ii) $x^2 = x$ for every invertible element $x \in A$.

Prove that:

(a) $a + x$ is not invertible for any $a, x \in A$, where a invertible and $x \neq 0$ is not invertible;

(b) $x^2 = x$ for all $x \in A$.

Originally Problem 4 from Grade 12 competition of the District Round of the 2017 Romania Math Olympiad.

We received 1 *submission. We present the solution by the Missouri State University Problem Solving Group.*

We claim that condition (a) holds regardless of whether A is a division ring and condition (b) does *not* follow from conditions (i) and (ii). We replace condition (i) with one that will (along with condition (ii)) imply condition (b).

If x is invertible and $x^2 = x$, then $x = 1$. Therefore the only invertible element of A is 1. If a is invertible, then $a = 1$ and if $x \neq 0$ (whether invertible or not), $a + x = 1 + x \neq 1$ and hence $a + x$ is not invertible, so condition (a) is satisfied.

Let $A = \mathbb{F}_2[t]$. Clearly A is not a division ring and the only invertible element of A is the only invertible element in $\mathbb{F}_2$, namely 1, which satisfies $1^2 = 1$. However, $t^2 \neq t$.

If the additional criterion that A is finite is added, then condition (b) follows from condition (ii) alone.

The characteristic subring of A must be isomorphic to $\mathbb{F}_2$, otherwise -1 would be an invertible element distinct from 1. Clearly $0^2 = 0$ and $1^2 = 1$. Choose any element $x \in A, x \neq 0, 1$. The subring of A generated by x and 1 is isomorphic to $\mathbb{F}_2[t]/(f(t))$ (via the map sending t to x), where $f(t)$ is a polynomial of degree greater than 1 (we need the fact that A is finite to guarantee that $f(t)$ is not the zero polynomial). Let

$$f(t) = \prod_{i=1}^{k} p_i(t)^{m_i}$$

be the factorization of $f(t)$, where $p_i(t) \neq p_j(t)$ for $i \neq j$. By the Chinese Remainder Theorem,

$$\mathbb{F}_2[t]/(f(t)) \cong \prod_{i=1}^{k} \mathbb{F}_2[t]/(p_i(t)^{m_i}).$$

If $m_i > 1$ for any i, then $1 - p_i(t)$ is a non-trivial unit in $\mathbb{F}_2[t]/(p_i(t)^{m_i})$, which gives a non-trivial unit in $\mathbb{F}_2[t]/(f(t))$, so this cannot occur and $m_i = 1$ for all i. If $\deg(p_i) > 1$ for any i, then

$$\mathbb{F}_2[t]/(p_i(t)^{m_i}) = \mathbb{F}_2[t]/(p_i(t))$$

is a field of order greater than 2 and hence has non-trivial units. Therefore, $f(t)$ must be a product of distinct linear factors. But there are only two linear polynomials over $\mathbb{F}_2$, so $f(t) = t(t-1)$ and hence $x^2 = x$.

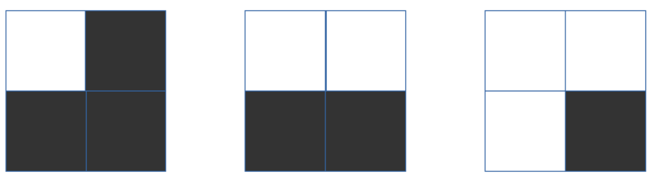
We end by classifying all finite rings A satisfying condition (ii). Since $x^2 = x$ for all $x \in A$, $a + b = (a + b)^2 = a^2 + ab + ba + b^2 = a + ab + ba + b$. Therefore $ab = -ba = ba$ (since the characteristic of A is 2) and A is a commutative ring. All finite commutative rings are products of local rings. If any of these local rings is not isomorphic to $\mathbb{F}_2$, we will have a non-trivial unit leading to a contradiction. Hence $A \cong \mathbb{F}_2^k$.

OC405. Each cell of a 100×100 table is painted either black or white and all the cells adjacent to the border of the table are black. It is known that in every 2×2 square there are cells of both colours. Prove that in the table there is a 2×2 square that is coloured in the chessboard manner.

Originally Problem 8 from Grade 9 competition of the 2017 Russia Math Olympiad.

We received 1 *submission. We present the solution by Oliver Geupel.*

We prove the property for an $m \times n$ table where m and n are even numbers greater than 2. The proof is by contradiction. Assume there is no 2×2 chessboard. Then, up to rotation about the centre, there are only the following three types of 2×2 squares:



Consider sides of cells that separate a black cell from a white one. Those sides can be arranged in disjoint closed nonintersecting lattice paths. Traverse any such lattice path, starting at a lattice point and arriving at the same point. Write L, R, U, D every time you pass a single side to the left, right, up and down, respectively. For a closed lattice path, the number of L's is the same as the number of R's, and the number of U's is equal to the number of D's. Hence, the lattice path consists of an even number of sides. Therefore, every such lattice path connects an even number of lattice points. As a consequence, the total number of lattice points that is part of any such lattice path is even.

Since in every 2×2 square there are cells of both colours, every interior lattice point of the table belongs to exactly one of our lattice paths. But the number of interior lattice points is $(m-1) \times (n-1)$, which is an odd number. This is the desired contradiction, which completes the proof.

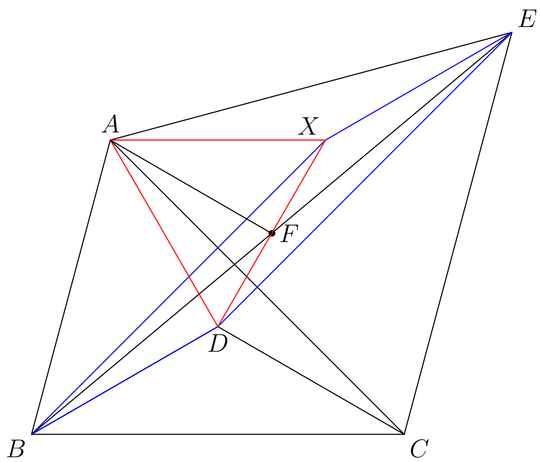
OC406. Let D be a point inside the triangle ABC such that $BD = CD$ and $\angle BDC = 120^\circ$. Let E be a point outside the triangle ABC such that $AE = CE$, $\angle AEC = 60^\circ$ and points B and E are in different half-planes with respect to AC. Prove that $\angle AFD = 90^\circ$, where F is the midpoint of the segment BE.

Originally Problem 2 from Grade 11 competition of the 2017 Moscow Math Olympiad.

We received 5 *submissions. We present two solutions.*

Solution 1, by Sushanth Sathish Kumar.

Let X be a point such that $DBXE$ is a parallelogram. Then, F is the midpoint of XD. Therefore, it is enough to show that A lies on the perpendicular bisector of XD, or that $AX = AD$.



Note that $AE = AC$, and $XE = BD = DC$, by construction. We claim that Ψ, the 60° rotation about A, maps D to X. Note that this will imply $AX = AD$, and we will be done. Clearly, under Ψ, C maps to E, since $\angle CAE = 60^\circ$. To show D maps to X, note that the angle formed by lines CD and EX is the same as the angle formed by lines CD and DB, which is $180^\circ - \angle BDC = 60^\circ$. Hence, the claim is proven, and we are done.

Remark. Note that the above proof additionally shows that $\angle DAX = 60^\circ$, which implies that triangle AFD is actually a 30-60-90 triangle, with the 30° angle at A.

Solution 2, by Ivko Dimitrić

We use a combination of vectors and complex numbers. Consider the given triangle ABC in the plane of complex numbers with vertices labeled counterclockwise and point D at the origin. Represent each point by a complex number denoted by the same capital letter. Each vector $\overrightarrow{XY}$ is represented by a complex number $Y - X$. In particular, $\overrightarrow{AC} = C - A$. Since $\overrightarrow{DC}$ is the rotation of $\overrightarrow{DB}$ through the angle $2\pi/3$, we have $C = B\,e^{i(2\pi/3)}$. Since $\triangle ACE$ is isosceles $(EA = EC)$ with vertex angle of 60° at E, it is, in fact, equilateral and $\angle EAC = 60^\circ$. Hence, $\overrightarrow{AE}$ is the result of rotation of $\overrightarrow{AC}$ through $\pi/3$. Hence,

$$E - A = (C - A)\,e^{i(\pi/3)} \quad \Longrightarrow \quad E = A + (C - A)\,e^{i(\pi/3)}.$$

Then,

$$\begin{aligned} 2\overrightarrow{DF} &= \overrightarrow{DB} + \overrightarrow{DE} \\ &= B + E \\ &= B + A + (C - A)e^{i(\pi/3)} \\ &= B + A + (Be^{i(2\pi/3)} - A)\,e^{i(\pi/3)} \\ &= A(1 - e^{i(\pi/3)}) \end{aligned}$$

Similarly,

$$\begin{aligned}
2\overrightarrow{AF} &= \overrightarrow{AB} + \overrightarrow{AE}\\
&= (B - A) + (E - A)\\
&= B - A + (C - A)e^{i(\pi/3)}\\
&= B - A + (Be^{i(2\pi/3)} - A)\, e^{i(\pi/3)}\\
&= -A(1 + e^{i(\pi/3)}).
\end{aligned}$$

The dot product of two vectors, represented by complex numbers $v = 2\overrightarrow{DF}$ and $w = 2\overrightarrow{AF}$ is $\langle v, w\rangle = \frac{1}{2}(v\bar{w} + \bar{v}w)$, which is zero if and only if the two vectors are perpendicular. In our case,

$$\begin{aligned}
\langle v, w\rangle &= \frac{1}{2}\Big[A(1 - e^{\frac{\pi}{3}i})(-\bar{A})(1 + e^{-\frac{\pi}{3}i}) + \bar{A}(1 - e^{-\frac{\pi}{3}i})(-A)(1 + e^{\frac{\pi}{3}i})\Big]\\
&= -\frac{1}{2}\, A\bar{A}\left(1 + e^{-\frac{\pi}{3}i} - e^{\frac{\pi}{3}i} - e^{0} + 1 + e^{\frac{\pi}{3}i} - e^{-\frac{\pi}{3}i} - e^{0}\right)\\
&= -\frac{1}{2}\, A\bar{A}\cdot 0\\
&= 0.
\end{aligned}$$

Hence, vectors $\overrightarrow{AF}$ and $\overrightarrow{DF}$ are perpendicular i. e. $\angle AFD = 90^\circ$.

OC407. The acute isosceles triangle ABC ($AB = AC$) is inscribed in a circle with center O. The rays BO and CO intersect the sides AC and AB in the points B' and C', respectively. A line l parallel to the line AC passes through point C'. Prove that the line l is tangent to the circumcircle ω of the triangle $B'OC$.

Originally Problem 3 from Grade 10 competition of the 2017 Russia Math Olympiad.

We received 5 *submissions. We present the solution by Sushanth Sathish Kumar.*

Let X be the reflection of A over $B'C'$. Given concyclic points P_1,P_2,...,P_n, let $(P_1P_2...P_n)$ denote the circle passing through them.

By construction A, O, X are collinear and X lies on l. We claim that l is tangent to $(B'OC)$ at X. Note that O is the circumcenter of triangle $XB'C'$. Indeed, the homothety at O taking BC to $B'C'$ also takes X to A, since $XC'||AC$ and $XB'||AB$. The same argument shows triangles OXC' and OAC are similar. Invoking symmetry,

$$\angle OXC' = \angle OAC = \angle OCA = \angle OBA = \angle OBC',$$

which shows that X lies on $(B'OC)$. To show tangency, note that

$$\angle B'XO = \angle OXC' = \angle XC'O = \angle XBO = \angle XBB',$$

and we are done.

OC408. Does there exist an infinite increasing sequence $a_1, a_2, a_3, \ldots$ of positive integers such that the sum of any two distinct terms of the sequence is coprime with the sum of any three distinct terms of the sequence?

Originally Problem 4 from Grade 9 competition of the 2017 Moscow Math Olympiad.

We received 1 *submission. We present the solution by Oliver Geupel.*

The answer is yes. We are going to prove that the sequence with $a_1 = 7$ and $a_n = (3a_{n-1})! + 1$ for $n > 1$ has the desired property. Note that all members of the sequence are odd. Also, for every $n \geq 2$ and integers u, v with the property $1 \leq v \leq 3a_{n-1}$, it holds that $a_n \equiv 1 \pmod{v}$; whence

$$(u,\ v) = (u + 1 - a_n,\ v).$$

We have to show that, for any indices i, j, k, ℓ, m with the property $i < j < k$ and $\ell < m$, it holds that $(a_i + a_j + a_k,\ a_\ell + a_m) = 1$.

First, consider the case where $|\{i, j, k, \ell, m\}| = 3$. Let a, b, c be members of (a_n) such that $a < b < c$. Then,

$$\begin{aligned}
&(a+b+c,\ a+b) = (a+b+1,\ a+b) = 1,\\
&(a+b+c,\ a+c) = (b,\ a+c) = (b,\ a+1) = (1,\ a+1) = 1,\\
&(a+b+c,\ b+c) = (a,\ b+c) = (a,\ b+1) = (a,\ 2) = 1,
\end{aligned}$$

which completes the case $|\{i, j, k, \ell, m\}| = 3$.

Next, consider the case where $|\{i, j, k, \ell, m\}| = 4$. Let a, b, c, d be members of (a_n) such that $a < b < c < d$. Then,

$$\begin{aligned}
&(a+b+c,\ a+d) = (a+b+c,\ a+1) = (a+2,\ a+1) = 1,\\
&(a+b+c,\ b+d) = (a+b+c,\ b+1) = (a+b+1,\ b+1) = (a,\ b+1) = 1,\\
&(a+b+c,\ c+d) = (a+b+c,\ c+1) = (a+b-1,\ c+1) = (a+b-1,\ 2) = 1,\\
&(a+b+d,\ a+c) = (a+b+1,\ a+c) = (a+2,\ a+1) = 1,\\
&(a+b+d,\ b+c) = (a+b+1,\ b+c) = (a+b+1,\ b+1) = 1,\\
&(a+b+d,\ c+d) = (a+b+d,\ c-a-b) = (a+b+1.\ 1-a-b) = 1,\\
&(a+c+d,\ a+b) = (a+2,\ a+1) = 1,\\
&(a+c+d,\ b+c) = (a+c+1,\ b+c) = (a+c+1,\ b-a-1) = (a+2,\ a) = 1,\\
&(a+c+d,\ b+d) = (a-b+1,\ b+1) = (a+2,\ b+1) = (a+2,\ 2) = 1,\\
&(b+c+d,\ a+b) = (b+2,\ a+b) = (b+2,\ a-2) = (3,\ a-2) = 1,\\
&(b+c+d,\ a+c) = (b+c+1,\ a+c) = (b+1-a,\ a+1) = (b+2,\ a+1) = 1,\\
&(b+c+d,\ a+d) = (b+c-a,\ a+1) = (2-a,\ a+1) = 1,
\end{aligned}$$

which completes the case $|\{i, j, k, \ell, m\}| = 4$.

Finally, consider the case where $|\{i, j, k, \ell, m\}| = 5$. Let a, b, c, d, e be members of (a_n) such that $a < b < c < d < e$. For every arrangement of the numbers

a, b, c, d, e as a sum of three and a sum of two numbers, we can successively reduce the numbers e, d, c, b, a in the gcd to 1, obtaining $(3, 2) = 1$, for example:

$$(a+d+e,\ b+c) = (a+d+1,\ b+c) = (a+2,\ b+c) = (a+2,\ 2) = (3,2) = 1.$$

This completes the proof that (a_n) has the required property.

OC409.

(a) Give an example of a continuous function $f : [0, \infty) \to \mathbb{R}$ such that

$$\lim_{x\to\infty} \frac{1}{x^2} \int_0^x f(t)\ dt = 1$$

and $f(x)/x$ has no limit as $x \to \infty$.

(b) Let $f : [0, \infty) \to \mathbb{R}$ be an increasing function such that

$$\lim_{x\to\infty} \frac{1}{x^2} \int_0^x f(t)\ dt = 1.$$

Prove that $f(x)/x$ has a limit as $x \to \infty$ and determine this limit.

Originally Problem 4 from Grade 12 competition of the 2017 Romania Math Olympiad.

We received 1 *submission. We present the solution by Omran Kouba.*

(a) First, consider $f(x) = x \sin x + 2x$. Clearly $f(x)/x$ has no limit as $x \to \infty$, while

$$\frac{1}{x^2} \int_0^x f(t)\, dt = 1 + \frac{\sin x - x\cos x}{x^2}$$

and this tends to 1 as $x \to \infty$.

(b) Let

$$G(x) = \frac{1}{x^2} \int_0^x f(t) dt.$$

For $0 < u < v$, and because f is increasing we have

$$(v-u)f(u) \le \int_u^v f(t)dt \le (v-u)f(v),$$

or

$$f(u) \le \frac{v^2G(v) - u^2G^2(u)}{v-u} \le f(v),$$

Now, consider $\lambda > 1$, $x > 0$ and apply the previous inequality with $(u, v) = (x, \lambda x)$ and $(u, v) = (x/\lambda, x)$ we conclude that

$$f(x) \le x\frac{\lambda^2 G(\lambda x) - G^2(x)}{\lambda - 1}, \quad \text{and} \quad x\frac{G(x) - G^2(x/\lambda)/\lambda^2}{1 - 1/\lambda} \le f(x)$$

Equivalently

$$\frac{\lambda^2 G(x) - G^2(x/\lambda)}{\lambda(\lambda-1)} \leq \frac{f(x)}{x} \leq \frac{\lambda^2 G(\lambda x) - G^2(x)}{\lambda - 1}.$$

Recalling that $\lim_{x\to\infty} G(x) = 1$ we conclude that, for all $\lambda > 1$ we have

$$1 + \frac{1}{\lambda} \leq \liminf_{x\to\infty} \frac{f(x)}{x} \leq \limsup_{x\to\infty} \frac{f(x)}{x} \leq 1 + \lambda$$

But $\lambda > 1$ is arbitrary. So,

$$2 = \liminf_{x\to\infty} \frac{f(x)}{x} \leq \limsup_{x\to\infty} \frac{f(x)}{x} = 2,$$

i.e.

$$\lim_{x\to\infty} \frac{f(x)}{x} = 2$$

which is the desired conclusion.

OC410. Let $a_0, a_1, \ldots, a_{10}$ be integers such that $a_0 + a_1 + \cdots + a_{10} = 11$. Find the maximum number of distinct integer solutions to the equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{10} x^{10} = 1.$$

Originally Problem 1 of Category 3 of the 2017 Hungary Math Olympiad.

We received no submissions to this problem.