TEACHING PROBLEMS

No. 3

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Unique Teenage Factorization

Try it. Write down any two or three "teenages", not necessarily different, and multiply them together. Now break the product back down into its factors so that they can be rearranged into a product of the ages of teenagers. The result is unique.

For example, $14 \times 15 \times 17 = 3570$. If we try to identify factors, it may be evident that 35 is a factor and so we have 35×102. Breaking this down further we have 7×5 and $2 \times 3 \times 17$. These factors can be repackaged as $(2 \times 7) \times (3 \times 5) \times 17$. Note that it would not have mattered if we began by observing 2 or 5 or 10 was a factor instead, as ultimately the prime factorization is unique.

Let us consider the same idea in reverse. That is, given the product of the ages of a group of teenagers is 3570, find the ages of the teenagers. Indeed we could break 3570 down fully into prime factors and put them back together to make suitable ages. Alternatively, one can recognize properties like the divisibility by 10 (and hence, by 5) that necessitate the inclusion of age 15 among them. Likewise, the evident divisibility by 7 in this case ensures that there will be a 14 year old. The third age of 17 falls out through the division process.

You are encouraged to take a calculator and simply multiply a bunch of ages of teenagers together. Then take this product apart to find the individual ages. This will enhance appreciation of the process. Both students and teachers will realize how easy it becomes to generate different examples, thus enabling people to try their own problems at a suitable pace or engage peers with fresh challenges. Here is an example for you to try:

> The product of the ages of a group of teenagers is 10584000. Find the ages of the teenagers.

Another teaching point that can be offered here concerns the idea of lower and upper bounds. Informally these concepts can be considered through attention to a different matter. The focus can be placed on the number of teenagers in the group rather than the specific ages. Keep in mind that we require a value of n for which the product lies between 12^n and 20^n. In fact, using powers of 10 rather than 12 can provide a ballpark figure quite quickly. Reverting to our earlier example with 3570, we can readily see that $10^3 < 3570 < 20^3$. In fact, it can be verified that $n = 3$ when powers of 12 are used also. So in the problem with 10584000 or a little more than 10^7, it seems possible that there may be as many as seven teenagers. However, checking we find that 12^7 exceeds 35 million and there are only six teenagers.

Looking ahead...

The idea underlying *Teaching Problems* is to highlight problems that teachers have found to be particularly valuable. It may be that they illustrate features of mathematics. Some problems lend themselves to multiple solutions or approaches that vary widely. Submissions of your examples of teaching problems with accompanying commentary are welcomed. Send them along please.

Problem solvers enjoy solving problems. In anticipation of future issues of *Teaching Problems*, a trio of problems is offered here for your consideration. Discussion of them will appear in the coming months. Experience with trying these problems may enrich the reading experience in future, while adding to the discussion. Comments on the problems before or after that time are welcomed.

The Ruler Problem

An unmarked ruler is known to be exactly 6 cm in length. It is possible to exactly measure all integer lengths from 1 cm to 6 cm using only two marks, at 1 cm and 4 cm, since $2 = 6 - 4, 3 = 4 - 1$, and $5 = 6 - 1$.

Determine the smallest number of marks required on an unmarked ruler 30 cm in length to exactly measure all integer lengths from 1 cm to 30 cm.

A Geometry Problem inviting Multiple Approaches

Given square $ABCD$, with E the midpoint of CD and F the foot of the perpendicular from B to AE, show that $CF = CD$.

A Handshake Problem with a Twist

Mr. and Mrs. Smith were at a party with three other married couples. Since some of the guests were not acquainted with one another, various handshakes took place. No one shook hands with his or her spouse, and of course, no one shook their own hand! After all of the introductions had been made, Mrs. Smith asked the other seven people how many hands each shook. Surprisingly, they all gave different answers. How many hands did Mr. Smith shake?