MATHEMATTIC

No. 6

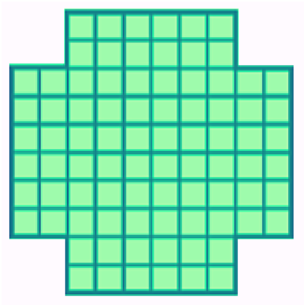
The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **September 30, 2019**.

MA26. Nine (not necessarily distinct) 9-digit numbers are formed using each digit 1 through 9 exactly once. What is the maximum possible number of zeros that the sum of these nine numbers can end with?

MA27. You want to play Battleship on a 10×10 grid with 2×2 squares removed from each of its corners:



What is the maximum number of submarines (ships that occupy 3 consecutive squares arranged either horizontally or vertically) that you can position on your board if no two submarines are allowed to share any common side or corner?

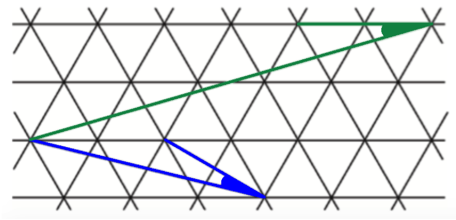
MA28. Prove that for all positive integers n, the number

$$\frac{1}{3}\left(4^{4n+1} + 4^{4n+3} + 1\right)$$

is not prime.

MA29. Find all positive integers n satisfying the following condition: numbers $1, 2, 3, \ldots, 2n$ can be split into pairs so that if numbers in each pair are added and all the sums are multiplied together, the result is a perfect square.

MA30. Consider the two marked angles on a grid of equilateral triangles.



Prove that these angles are equal.

. . . .

Les problémes dans cette section sont appropriés aux étudiants de l'école secondaire.

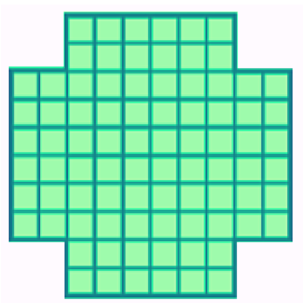
Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 septembre 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA26. Neuf nombres à 9 chiffres sont formés, chacun se servant des chiffres de 1 à 9 une seule fois. Ces neuf nombres n'ont pas besoin d'être distincts. Quel est le nombre maximal de zéros pouvant se retrouver à la fin de la représentation décimale de la somme de ces neuf nombres?

MA27. Un jeu de bataille navale se tient sur un grillage 10×10, duquel on a enlevé les cases 2×2 de chacun des coins.



Étant donné qu'un sousmarin occupe 3 cases consécutives, horizontalement ou verticalement, quel est le nombre maximal de sousmarins qu'on puisse placer sur le grillage spécial de façon à ce que les sousmarins ne partagent jamais un coin ou un côté?

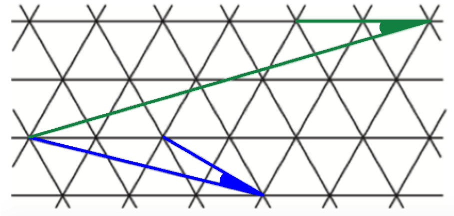
MA28. Démontrer que pour tout entièrs positifs n, le nombre

$$\frac{1}{3}\left(4^{4n+1}+4^{4n+3}+1\right)$$

n'est pas premier.

MA29. Déterminer tous les entièrs positifs n tels que si les nombres $1, 2, 3, \ldots, 2n$ sont regroupés en paires de façon bien choisie, que la somme de chaque paire est calculée et que ces sommes sont multipliées, le résultat est un carré parfait.

MA30. Considérer les deux angles situés sur un grillage de triangles équilatéraux.



Démontrer que ces angles sont égaux.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(1), p. 4–5.

MA1. How many two-digit numbers are there such that the difference of the number and the number with the digits reversed is a non-zero perfect square? Problem extension: What happens with three-digit numbers? four-digit numbers?

Originally Question 7 from the 1999 W.J. Blundon Contest.

We received 5 submissions of which one was correct and complete. We present the solution by Sophie Bekerman (Los Gatos High School), modified by the editor.

Two-digit numbers.

Let A be such a two digit number. We can express A as $10x + y$ where x is the first digit, y is the second digit, and $x, y \in [0, \ldots, 9]$. Let $\bar{A}$ be A with the digits reversed, note that $\bar{A} = 10y + x$. Given

$$A - \bar{A} = 10x + y - (10y + x) = 9x - 9y,$$

let $9x - 9y = a^2$ where $a \in \mathbb{N}$. It follows that

$$9x - 9y = a^2 \Leftrightarrow x - y = \left(\frac{a}{3}\right)^2$$

and $\left(\frac{a}{3}\right)^2 \in \mathbb{N}$ since it is the difference of two natural numbers. $\left(\frac{a}{3}\right)^2 \leq 9$ since $x - y \leq 9$. The only perfect squares that meet these conditions are 1, 4, and 9. Therefore, the differences of the digits of A are 1, 4, or 9. If $x - y = n$, their difference can be written as $(n + k) - k$ where $n + k = x$ and $k = y$. Since

$$n + k \leq 9 \Leftrightarrow k \leq 9 - n,$$

k can take any value from 0 to $9 - n$. In total, there are $10 - n$ ways to represent each difference. As $n \in [1, 4, 9]$, there are

$$(10 - 1) + (10 - 4) + (10 - 9) = 16$$

possible values of A.

Three-digit numbers.

Let B be such a three digit number. We can express B as $100x + 10y + z$, where x is the first digit, y is the second digit, z is the third digit, and $x, y, z \in [0, \ldots, 9]$. Let $\bar{B}$ be B with the digits reversed, note that $\bar{B} = 100z + 10y + x$. Given

$$B - \bar{B} = 100x + 10y + z - (100z + 10y + x) = 99x - 99z,$$

let $99\,(x-z)=b^2$ where $b\in\mathbb{N}$. It follows that

$$99\,(x-z)=b^2 \Leftrightarrow 11\,(x-z)=\left(\frac{b}{3}\right)^2$$

and $\left(\frac{b}{3}\right)^2\in\mathbb{N}$ since it is the difference of two natural numbers. $\left(\frac{b}{3}\right)^2\leq 9$ since $x-z\leq 9$. For $11\,(x-z)$ to be a perfect square, $(x-z)$ has to be a factor of 11. This is impossible since $x-z\leq 9$. Therefore, there are no possible forms of B.

Four-digit numbers.

Let C be such a four digit number. We can express C as $1000w+100x+10y+z$, where w is the first digit, x is the second digit, y is the third digit, z is the fourth digit, and $w,x,y,z\in[0,\ldots,9]$. Let $\bar{C}$ be C with the digits reversed, note that $\bar{C}=1000z+100y+10x+w$. Given

$$\begin{aligned} C-\bar{C} &= 1000w+100x+10y+z-(1000z+100y+10x+w)\\ &= 999w+90x-90y-999z \end{aligned}$$

let $999\,(w-z)+90\,(x-y)=c^2$ where $c\in\mathbb{N}$. It follows that

$$999\,(w-z)+90\,(x-y)=c^2,$$

or, equivalently,

$$111\,(w-z)+10\,(x-y)=\left(\frac{c}{3}\right)^2.$$

If $w=z\Leftrightarrow w-z=0$ then that leaves $10\,(x-y)=\left(\frac{c}{3}\right)^2$. For $10\,(x-y)$ to be a perfect square, $(x-y)$ has to be a factor of 10, which is impossible since $x-y\leq 9$. Therefore, $w-z\neq 0$ and $111\leq\left(\frac{c}{3}\right)^2\leq 1089$.

The perfect squares between 111 and 1089 are 121, 484, 576, 625, 676, and 1089. These are found simply by searching through every perfect square in the range $[111,1089]$ and seeing if the perfect square can be expressed in the form $111m+10n$, where $m,n\in\mathbb{N}$. The only case where $x<y$ is $576=111\cdot 6-10\cdot 9$. For all the other possible squares, $x-y$ happens to be positive.

For 576, $w-z=6$ and $x-y=-9$. There are $10-6=4$ possible pairs of w and z that yield a difference of 6. There is $10-9=1$ possible pair of x and y that yield a difference of 9. Therefore there are 4 possible combinations of w,x,y, and z that will yield 576.

This same methodology applies to the other possible perfect squares for a total of

$$\begin{aligned} &(10-1)\cdot(10-1)+(10-4)\cdot(10-4)+(10-6)\cdot(10-9)+\\ &\quad(10-5)\cdot(10-7)+(10-6)\cdot(10-1)+(10-9)\cdot(10-9)=173 \end{aligned}$$

combinations of C.

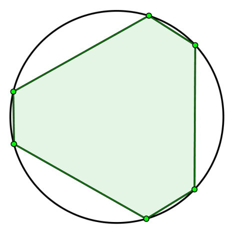
MA2. A sequence $t_1, t_2, \ldots$ beginning with any two positive numbers is defined such that for $n > 2$, $t_n = \dfrac{1 + t_{n-1}}{t_{n-2}}$. Show that such a sequence must repeat itself with a period of 5.

Originally Question 9 from the 2002 W.J. Blundon Contest.

We received 5 solutions. We present the solution by Richard Hess.

Start with a and b. Then the next terms are $(1+b)/a$, $(a+b+1)/(ab)$, $(a+1)/b$, a, b, $\ldots$. This sequence has a period of five since terms six and seven duplicate terms one and two.

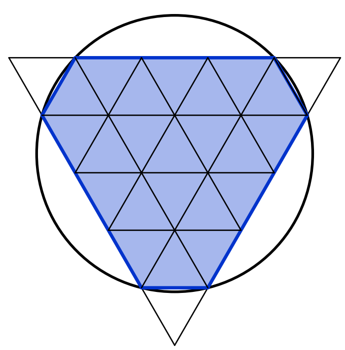
MA3. A hexagon H is inscribed in a circle, and consists of three segments of length 1 and three segments of length 3. Find the area of H.



Originally Question 10 from the 2000 W.J. Blundon Contest.

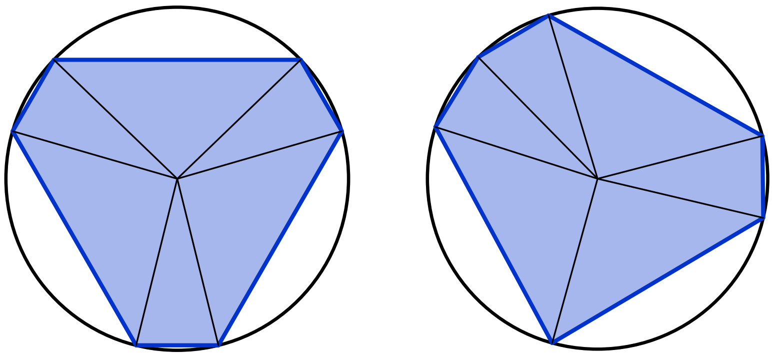
We received seven solutions, out of which we present the one by Valcho Milchev, lightly edited.

By symmetry, all the internal angles of the hexagon H are equal and thus 120°. This means that H may be tiled by equilateral triangles as shown in the figure:



H is composed of 22 equilateral triangles of side length 1, each of which has an area of $\dfrac{\sqrt{3}}{4}$. Therefore the area of H is $\dfrac{11\sqrt{3}}{4}$.

Editor's Comments. The statement of Problem MA3 did not specify in which order the segments appear in the hexagon, even though the picture suggested a specific arrangement. However, it turns out that all cyclic hexagons with three edges of length 1 and three edges of length 3 have the same area. This can be seen by drawing the radii from the centre of the circle to the six vertices of the hexagon (see figure below). This splits the hexagon into six isosceles triangles with leg lengths equal to the radius of the circles. Three of the isosceles triangles have base length 3 and three have base length 1, irrespective of the arrangement of the edges in the hexagon.



MA4. For what conditions on a and b is the line $x + y = a$ tangent to the circle $x^2 + y^2 = b$?

Originally Question 9 from the 2002 W.J. Blundon Contest.

We received seven submissions, all of which were correct and complete. We present the joint solution by Amit Kumar Basistha (Anundoram Borooah Academy High School) and Sophie Bekerman (Los Gatos High School), done independently, slightly modified by the editor.

$x + y = a$ is tangent to $x^2 + y^2 = b$ when the system

$$\begin{cases} x + y = a \\ x^2 + y^2 = b \end{cases}$$

has exactly one solution. Given that $x + y = a \Leftrightarrow y = a - x$, by substitution we see that

$$x^2 + y^2 = b \Leftrightarrow x^2 + (a - x)^2 = b \Leftrightarrow 2x^2 - 2ax + a^2 - b = 0$$

A quadratic equation has one solution if and only if the discriminant is equal to 0. By construction, our expression has only one solution, thus by setting the discriminant Δ of the above expression to 0 we see that

$$\Delta = 4a^2 - 4\left(2\left(a^2 - b\right)\right) = 0 \Leftrightarrow a^2 = 2b$$

When there is one solution for x, there is also only one solution for y since $y = a - x$. Hence, when $a^2 = 2b$, the line $x + y = a$ is tangent to the circle $x^2 + y^2 = b$.

MA5. Point P lies in the first quadrant on the line $y = 2x$. Point Q is a point on the line $y = 3x$ such that PQ has length 5 and is perpendicular to the line $y = 2x$. Find the point P.

Originally Question 8 from the 2002 W.J. Blundon Contest.

We received 5 submissions of which 3 were correct and complete. We present the solution by Vitthal Ingle and Konstantine Zelator, done independently.

Let θ_1, θ_2 be the angles made between the x-axis and the lines $y = 3x$ and $y = 2x$ respectively. Clearly, $\tan\theta_1 = 3$ and $\tan\theta_2 = 2$. Let $\alpha = \theta_1 - \theta_2$, the angle between the lines $y = 2x$ and $y = 3x$. By the angle subtraction identity for tangent:

$$\tan\alpha = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} = \frac{3-2}{1+3\cdot 2} = \frac{1}{7}.$$

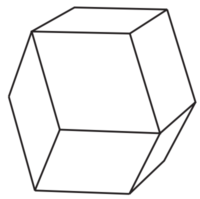
Let O be the origin. We have

$$\tan\alpha = \frac{1}{7} = \frac{\overline{PQ}}{\overline{OP}} = \frac{5}{\overline{OP}},$$

and so $\overline{OP} = 35$. Let P have coordinates $(a, 2a)$, and let M be the projection of P onto the x-axis. Now $\overline{OP}^2 = \overline{OM}^2 + \overline{MP}^2$, and so $35^2 = a^2 + 4a^2 = 5a^2$. It follows that $a = 7\sqrt{5}$ and so P has coordinates $(7\sqrt{5}, 14\sqrt{5})$. Note that the solution $a = -7\sqrt{5}$ gives a point in the third quadrant, and so can not be the answer.

CONTEST CORNER SOLUTIONS

CC339. A *rhombic dodecahedron* has twelve congruent rhombic faces; each vertex has either four small angles or three large angles meeting there. If the edge length is 1, find the volume in the form $\dfrac{p+\sqrt{q}}{r}$, where p, q, and r are natural numbers and r has no factor in common with p or q.

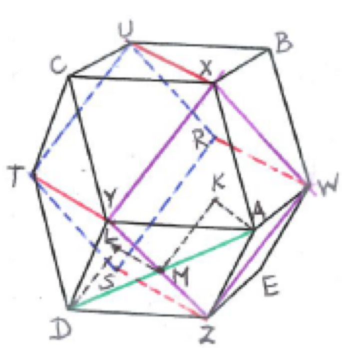


Originally from the 2018 Science Atlantic Math Competition.

The statement of this problem originally appeared in ***Crux*** *44(8). We received no solutions to this problem for the original publication. We have since received a solution by Ivko Dimitrić showing that the statement of the problem is not correct. We present the solution here.*

We have $f = 12$ rhombic faces, so the number of edges is $e = \frac{12\cdot 4}{2} = 24$. By Euler's formula, $v - e + f = 2$, we find the number of vertices to be $v = 14$. Moreover, if x is the number of vertices of degree 4 and y the number of those of degree 3, then from $x + y = 14$ and $e = \frac{4x+3y}{2} = 24$ we get $x = 6$ and $y = 8$.

It is a general fact that a rhombic dodecahedron is a semi-regular polyhedron with symmetries and transitive faces, which means that for each pair α, β of faces there is an isometry of the solid that takes α to β. As a consequence of the symmetries of this polyhedron, the arrangement of edges, the measures of angles in the same position and geometric picture about one vertex is exactly the same as the situation about another vertex of the same degree.



We use the labeling of vertices and points as shown on the diagram. Because of the symmetry, there exists an axis through a selected degree-4 vertex, which is equally inclined to each of the four edges incident to that vertex and there exists an angle so that the rotation through that angle about the axis will take each edge from that vertex to the next one in cyclic order and hence each vertex of the quadrilateral $WXYZ$ to its neighbouring vertex in cyclic order. Thus, these four vertices lie in the same plane perpendicular to the axis of rotation and in that plane each vertex is carried to its neighbour in the same sense by induced rotation in that plane through the same angle, so the quadrilateral is a square. Hence, each degree-4 vertex of the dodecahedron together with four neighboring vertices determines a pyramid with a square base formed by the four neighboring vertices, where the apex (a degree-4 vertex) is projected orthogonally to the center of the square base. Two square pyramids with apexes A and D are joined by the common edge YZ and the faces AYZ and DYZ that share that edge are two congruent triangular halves of rhomboid face $AZDY$ of the dodecahedron. Let a be the edge length of the polyhedron and h be the height of each of the six square pyramids with apex at a degree-4 vertex, such as pyramids $AWXYZ$ and $DTYZS$. The square bases of these six pyramids are congruent and their union forms the surface of a cube, since the dihedral angle at each edge is $90°$, for example $XY \perp XW$ and $XY \perp XU$, so XY is perpendicular to the square face $XURW$. We call this cube the core cube.

Let M be the midpoint of a shorter diagonal YZ of the rhombic face $AZDY$. Because of the congruence of the square pyramids with apexes A and D, the segments AM and DM are equally inclined to the corresponding bases of these pyramids and since the dihedral angle of the core cube at M is $90°$ and $\angle AMD = 180°$, it follows that $\angle KMA = \angle LMD = 45°$, i. e. $\triangle AKM$ and $\triangle DLM$ are two congruent isosceles right triangles and hence $s = h\sqrt{2}$, whereas the edge of the core cube is $2h$.

Then from (a half of) the isosceles $\triangle AYZ$ we get $(h\sqrt{2})^2 + h^2 = a^2$, which gives $h = a/\sqrt{3}$. The rhombic dodecahedron is constructed as the union of the cubical core of edge length $2h$ whose vertices are eight degree-3 vertices of the dodecahedron and six congruent square pyramids of height h, surmounting each of the six faces of the cube on the outside.Thus, its volume is

$$V = \left(\frac{2a}{\sqrt{3}}\right)^3 + 6 \cdot \frac{1}{3}\,(2h)^2 h = \frac{8a^3}{3\sqrt{3}} + 8\left(\frac{a}{\sqrt{3}}\right)^3 = \frac{16\sqrt{3}}{9}\,a^3.$$

When $a = 1$, the answer can be written as $\frac{0+\sqrt{768}}{9}$ which is the required expression, but it cannot be written in the form $\frac{p+\sqrt{q}}{r}$ where p, q, r are integers with r having no factor in common with p and q since 3 is the common factor of p, q, r.

Remark. Some nice pictures and additional information on rhombic dodecahedron can be found at `https://en.wikipedia.org/wiki/Rhombic.dodecahedron` and at `http://mathworld.wolfram.com/RhombicDodecahedron.html`.

PROBLEM SOLVING VIGNETTES

No.6

Shawn Godin

Repdigit Recreations

In this issue we will look at a couple of problems from the course C&O 380 that I took from Ross Honsberger that has been featured in previous columns. The next set of problems are:

#16. A is the integer $666\cdots 66$, containing 666 sixes. B is the integer $333\cdots 33$, containing 666 threes. State the value of AB.

#17. Show that a positive integer, with more than one digit, all of whose digits are the same, cannot be a perfect square.

#18. Show that the sum of the squares of 83 consecutive natural numbers is never a perfect square.

#19. Devise a method of trisecting a given line segment, using only straight-edge and compasses, which does not involve parallel lines.

#20. Construct an equilateral triangle so that it has one vertex on each of three given parallel lines.

Problems #16 and #17 both deal with *repdigit* numbers. That is, numbers that are comprised of a single digit repeated a number of times. Don Rideout, in problem #3 of his vignette [2019: 45(3), p. 120], looked at *repunit* numbers, that is, repdigit numbers made up of only the digit 1. We will run into a couple of the properties of these numbers as we solve the two problems. We will look at #17 first.

How do we know if a number is a perfect square? Looking at the first few squares we start to see a pattern in the units digit.

$1^2 = 1$	$2^2 = 4$	$3^2 = 9$	$4^2 = 16$	$5^2 = 25$
$6^2 = 36$	$7^2 = 49$	$8^2 = 64$	$9^2 = 81$	$10^2 = 100$
$11^2 = 121$	$12^2 = 144$	$13^2 = 169$	$14^2 = 196$	$15^2 = 225$
$16^2 = 256$	$17^2 = 289$	$18^2 = 324$	$9^2 = 361$	$20^2 = 400$

The unit digits follow the pattern

$$1, 4, 9, 6, 5, 6, 9, 4, 1, 0, 1, 4, 9, \ldots$$

If we use modular arithmetic, as in some recent columns, we would get the following:

$n \pmod{10}$	0	1	2	3	4	5	6	7	8	9
$n^2 \pmod{10}$	0	1	4	9	6	5	6	9	4	1

which tells us the same thing: the units digit of a perfect square is $0, 1, 4, 5, 6$, or 9. Hence the repdigit numbers $222\cdots22$, $333\cdots33$, $777\cdots77$, and $888\cdots88$ cannot be perfect squares.

We write the other repdigit numbers in the form

$$\begin{aligned} 111\cdots11 &= 1 \times 111\cdots11 \\ 444\cdots44 &= 4 \times 111\cdots11 \\ 555\cdots55 &= 5 \times 111\cdots11 \\ 666\cdots66 &= 6 \times 111\cdots11 \\ 999\cdots99 &= 9 \times 111\cdots11. \end{aligned}$$

Clearly $555\cdots55$ is not a perfect square since $5 \mid 555\cdots55$, but $5 \nmid 111\cdots11$. Similarly, $666\cdots66$ is not a perfect square.

The remaining three candidates are written as a perfect square times $111\cdots11$. Thus if $111\cdots11$ is a perfect square, then so is $444\cdots44$ and $999\cdots99$. If $111\cdots11$ is not a perfect square then neither are $444\cdots44$ and $999\cdots99$.

To determine if $111\cdots11$ is a perfect square we will go back to modular arithmetic and look at numbers modulo 4.

$n \pmod 4$	0	1	2	3
$n^2 \pmod 4$	0	1	0	1

So if a number is a perfect square it must be congruent to 0 or 1 modulo 4. Taking into account that $4 \mid 100$ and hence $4 \mid 10^n$ when $n \geq 2$ (so $10^n \equiv 0 \pmod 4$) we get

$$111\cdots11 \equiv 11 \equiv 3 \pmod 4$$

and so $111\cdots11$ is not a perfect square and therefore no repdigit number, of more than one digit, is a perfect square.

Next, we will look at problem #16. A few computations suggest a pattern:

$$\begin{aligned} 6 \times 3 &= 18 & 66 \times 33 &= 2\,178 \\ 666 \times 333 &= 221\,778 & 6\,666 \times 3\,333 &= 22\,217\,778 \\ 66\,666 \times 33\,333 &= 2\,222\,177\,778 & 666\,666 \times 333\,333 &= 222\,221\,777\,778 \end{aligned}$$

that is,

$$\overbrace{666\cdots66}^{n\ \text{6s}} \times \overbrace{333\cdots33}^{n\ \text{3s}} = \overbrace{222\cdots22}^{n-1\ \text{2s}}1\overbrace{777\cdots77}^{n-1\ \text{7s}}8. \tag{1}$$

It is one thing to see a pattern and be certain it is true. It is another thing to *prove* that the pattern does indeed hold. The pattern seems to call out for mathematical induction like we saw in the last issue [2019: 45(5), p. 236-240].

To make our lives easier, we will introduce the sequence of repunit numbers

$$\{U_n\}_{n=1}^{\infty} = \{1, 11, 111, 1111, \ldots\}.$$

Our proposition that we would like to prove is

$$P_n: \quad (6U_n)(3U_n) = 10^n[2U_n - 1] + 7U_n + 1. \tag{2}$$

You may want to convince yourself that (2) is equivalent to (1).

To aid us in our proof we will need the following properties of the repunit numbers:

$$10 \times U_n + 1 = U_{n+1} \tag{3}$$
$$U_a + 10^a \times U_b = U_{a+b} \tag{4}$$

We leave the proofs of these as exercises. Now on to our proof by induction.

If we look at P_1, we get

$$(6U_1)(3U_1) = 6 \times 3 = 18$$

and

$$10^1[2U_1 - 1] + 7U_1 + 1 = 10 \times (2 - 1) + 7 + 1 = 18$$

so the proposition is true for $n = 1$.

Suppose P_n is true for some $n = k \in \mathbb{N}$, then

$$(6U_k)(3U_k) = 10^k[2U_k - 1] + 7U_k + 1. \tag{5}$$

So, using (3) we get

$$\begin{aligned}(6U_{k+1})(3U_{k+1}) &= (6(10U_k + 1))(3(10U_k + 1)) \\ &= 100((6U_k)(3U_k)) + 360U_k + 18 \qquad (6)\end{aligned}$$

Combining (5) with (6) yields

$$\begin{aligned}(6U_{k+1})(3U_{k+1}) &= 100(10^k[2U_k - 1] + 7U_k + 1) + 360U_k + 18 \\ &= 10^{k+2}[2U_k - 1] + 1060U_k + 118 \qquad (7)\end{aligned}$$

Breaking the right side of (7) into two parts and using the properties (3) and (4), yields

$$\begin{aligned}10^{k+2}[2U_k - 1] &= 10^{k+1}[2(10U_k + 1 - 1) - 10] \\ &= 10^{k+1}[2(U_{k+1} - 1) - 10] \\ &= 10^{k+1}[2U_{k+1} - 12] \\ &= 10^{k+1}[2U_{k+1} - 1 - 11] \\ &= 10^{k+1}[2U_{k+1} - 1] - 11 \times 10^{k+1} \qquad (8)\end{aligned}$$

and

$$\begin{aligned}
1060U_k + 118 &= 1000U_k + 60U_k + 118 \\
&= 1000(U_{k-2} + 10^{k-2}U_2) + 60(U_2 + 10^2U_{k-2}) + 118 \\
&= 7000U_{k-2} + 11 \times 10^{k+1} + 778 \\
&= 7000U_{k-2} + 777 + 1 + 11 \times 10^{k+1} \\
&= 7U_{k+1} + 1 + 11 \times 10^{k+1} \qquad (9)
\end{aligned}$$

Putting (8) and (9) back into (7) yields

$$\begin{aligned}
(6U_{k+1})(3U_{k+1}) &= 10^{k+1}[2U_{k+1} - 1] - 11 \times 10^{k+1} + 7U_{k+1} + 1 + 11 \times 10^{k+1} \\
&= 10^{k+1}[2U_{k+1} - 1] + 7U_{k+1} + 1
\end{aligned}$$

which shows that P_{k+1} is true and completes the induction. So for the problem at hand we have

$$\overbrace{666\cdots66}^{666\ \text{6s}} \times \overbrace{333\cdots33}^{666\ \text{3s}} = \overbrace{222\cdots22}^{665\ \text{2s}}1\overbrace{777\cdots77}^{665\ \text{7s}}8.$$

It is nice to have an opportunity to use a new tool, but it is always nice to find a slick solution such as

$$\begin{aligned}
\overbrace{666\cdots66}^{666\ \text{6s}} \times \overbrace{333\cdots33}^{666\ \text{3s}} &= 6 \times 3 \times \overbrace{111\cdots11}^{666\ \text{1s}} \times \overbrace{111\cdots11}^{666\ \text{1s}} \\
&= 9 \times 2 \times \overbrace{111\cdots11}^{666\ \text{1s}} \times \overbrace{111\cdots11}^{666\ \text{1s}} \\
&= \overbrace{999\cdots99}^{666\ \text{9s}} \times \overbrace{222\cdots22}^{666\ \text{2s}} \\
&= (10^{666} - 1) \times \overbrace{222\cdots22}^{666\ \text{2s}} \\
&= \overbrace{222\cdots22}^{666\ \text{2s}}\overbrace{000\cdots00}^{666\ \text{0s}} - \overbrace{222\cdots22}^{666\ \text{2s}} \\
&= \overbrace{222\cdots22}^{665\ \text{2s}}1\overbrace{777\cdots77}^{665\ \text{7s}}8.
\end{aligned}$$

The biggest hammer isn't always the best tool for the job.

A little manipulation tells us that the repdigit number $\overbrace{ddd\cdots dd}^{n\ \text{ds}}$, where $d \in \{1, 2, \ldots, 9\}$ can be written as

$$\overbrace{ddd\cdots dd}^{n\ \text{ds}} = \frac{d}{9}\left(\overbrace{999\cdots99}^{n\ \text{9s}}\right) = \frac{d}{9}(10^n - 1)$$

which make sense as

$$\overbrace{ddd\cdots dd}^{n\text{ ds}} = d + 10d + 100d + \cdots + d \times 10^{n-1}$$

is a geometric series.

Enjoy the rest of the problems from the problem set, and you may enjoy the following problem from the vault:

> Find a quadratic polynomial $f(x)$ such that, if n is a positive integer consisting of the digit 5 repeated k times, then $f(n)$ consists of the digit 5 repeated $2k$ times. (For example, $f(555) = 555555$.)

This was Mayhem problem M256 that appeared in [2006: 32(5), p. 265-266] and the solution is in [2007: 33(5), p. 271] which can be generalized to

> Find a degree d polynomial $f(x)$ such that, if n is a positive integer consisting of the digit 5 repeated k times, then $f(n)$ consists of the digit 5 repeated dk times.

TEACHING PROBLEMS

No. 3

John McLoughlin

Unique Teenage Factorization

Try it. Write down any two or three "teenages", not necessarily different, and multiply them together. Now break the product back down into its factors so that they can be rearranged into a product of the ages of teenagers. The result is unique.

For example, $14 \times 15 \times 17 = 3570$. If we try to identify factors, it may be evident that 35 is a factor and so we have 35×102. Breaking this down further we have 7×5 and $2 \times 3 \times 17$. These factors can be repackaged as $(2 \times 7) \times (3 \times 5) \times 17$. Note that it would not have mattered if we began by observing 2 or 5 or 10 was a factor instead, as ultimately the prime factorization is unique.

Let us consider the same idea in reverse. That is, given the product of the ages of a group of teenagers is 3570, find the ages of the teenagers. Indeed we could break 3570 down fully into prime factors and put them back together to make suitable ages. Alternatively, one can recognize properties like the divisibility by 10 (and hence, by 5) that necessitate the inclusion of age 15 among them. Likewise, the evident divisibility by 7 in this case ensures that there will be a 14 year old. The third age of 17 falls out through the division process.

You are encouraged to take a calculator and simply multiply a bunch of ages of teenagers together. Then take this product apart to find the individual ages. This will enhance appreciation of the process. Both students and teachers will realize how easy it becomes to generate different examples, thus enabling people to try their own problems at a suitable pace or engage peers with fresh challenges. Here is an example for you to try:

> The product of the ages of a group of teenagers is 10584000. Find the ages of the teenagers.

Another teaching point that can be offered here concerns the idea of lower and upper bounds. Informally these concepts can be considered through attention to a different matter. The focus can be placed on the number of teenagers in the group rather than the specific ages. Keep in mind that we require a value of n for which the product lies between 12^n and 20^n. In fact, using powers of 10 rather than 12 can provide a ballpark figure quite quickly. Reverting to our earlier example with 3570, we can readily see that $10^3 < 3570 < 20^3$. In fact, it can be verified that $n = 3$ when powers of 12 are used also. So in the problem with 10584000 or a little more than 10^7, it seems possible that there may be as many as seven teenagers. However, checking we find that 12^7 exceeds 35 million and there are only six teenagers.

Looking ahead...

The idea underlying *Teaching Problems* is to highlight problems that teachers have found to be particularly valuable. It may be that they illustrate features of mathematics. Some problems lend themselves to multiple solutions or approaches that vary widely. Submissions of your examples of teaching problems with accompanying commentary are welcomed. Send them along please.

Problem solvers enjoy solving problems. In anticipation of future issues of *Teaching Problems*, a trio of problems is offered here for your consideration. Discussion of them will appear in the coming months. Experience with trying these problems may enrich the reading experience in future, while adding to the discussion. Comments on the problems before or after that time are welcomed.

The Ruler Problem

An unmarked ruler is known to be exactly 6 cm in length. It is possible to exactly measure all integer lengths from 1 cm to 6 cm using only two marks, at 1 cm and 4 cm, since $2 = 6 - 4, 3 = 4 - 1$, and $5 = 6 - 1$.

Determine the smallest number of marks required on an unmarked ruler 30 cm in length to exactly measure all integer lengths from 1 cm to 30 cm.

A Geometry Problem inviting Multiple Approaches

Given square $ABCD$, with E the midpoint of CD and F the foot of the perpendicular from B to AE, show that $CF = CD$.

A Handshake Problem with a Twist

Mr. and Mrs. Smith were at a party with three other married couples. Since some of the guests were not acquainted with one another, various handshakes took place. No one shook hands with his or her spouse, and of course, no one shook their own hand! After all of the introductions had been made, Mrs. Smith asked the other seven people how many hands each shook. Surprisingly, they all gave different answers. How many hands did Mr. Smith shake?