The Orthocentric Distances

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In this article, I introduce some distances related to the orthocenter, and their applications in the resolutions of challenging geometry problems. The most curious application of the distances is an alternate proof of the existence of the nine point circle in a triangle. If you are unfamiliar with the classic proof of this theorem, trying to prove all nine points are concyclic might seem like a nightmare. In Section 3, I give an intuitive proof of it using the results developed in Section 2.

1 What are the distances?

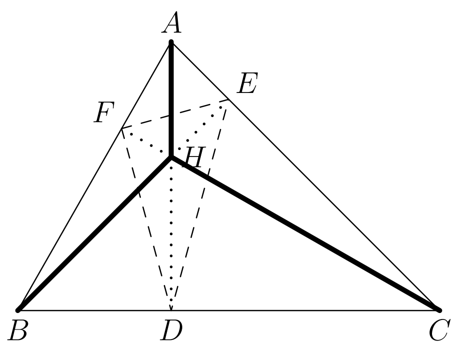
Theorem 1.1 (Orthocentric Distances, [3]) *Let ABC be any triangle with orthocenter H. Denote by D, E, F the feet of the A-, B-, C- altitudes, respectively. If we invoke the notion of directed lengths, then the following hold:*

a) $HA = 2R\cos A$,

b) $HD = 2R\cos B\cos C$,

c) $EF = a\cos A$

We invite the reader to prove the theorem. Note that invoking the notion of directed lengths does not restrict the distances to just acute triangles. Pictorially, the lemma gives these lengths,



It is worth noting a few facts that follow immediately.

Corollary 1.2 *In $\triangle ABC$ with orthocenter H and feet of altitudes D, E, F, we have $(HA)(HD) = (HB)(HE) = (HC)(HF)$.*

Corollary 1.3 *The power of orthocenter H with respect to the circumcircle of $\triangle ABC$ is $8R^2 \cos A\cos B\cos C$.*

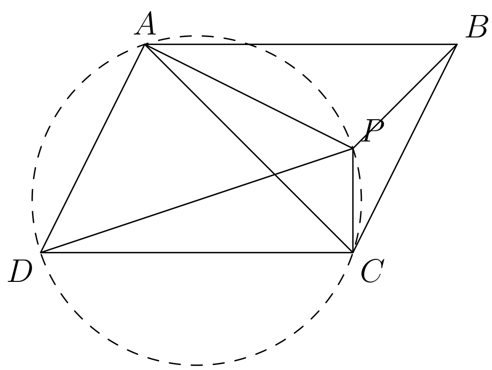
Corollary 1.3 follows from noting that the reflection of H over the sides of $\triangle ABC$ lie on its circumcircle. Looking back at the diagram, we see that A is the orthocenter of HBC, B is the orthocenter of HCA and that C is the orthocenter of HAB. For this reason, the points A, B, C, H are said to form an *orthocentric tetrad*.

Remark *The formula $AD = bc/2R$ is much more useful than the "areal" length of AD for problems using these distances.*

2 Examples

Here we will present instructive problems that succumb to the formulas described.

Example 2.1 (CMIMC 2016) In parallelogram $ABCD$, angles B and D are acute while angles A and C are obtuse. The perpendicular from C to AB and the perpendicular from A to BC intersect at a point P inside the parallelogram. If $PB = 700$ while $PD = 821$ what is AC?



Solution. Perhaps the only difficulty in solving this problem is finding the right triangle to apply our lemma on. A natural candidate is $\triangle ABC$ since two of its altitudes are already drawn in. Letting R be the circumradius, our formulas give

$$PB = 2R\cos B, PC = 2R\cos C, PA = 2R\cos A.$$

The only length we need now is PD. But we can easily get this from Ptolemy's Theorem on $PADC$! That gives

$$(PD)(AC) = (PA)(CD) + (PC)(AD),$$

and plugging in the distances, we see the right hand side is just $2Rb$. Hence $2R = 821$. Also,

$$PB = 2R\cos B = 700,$$

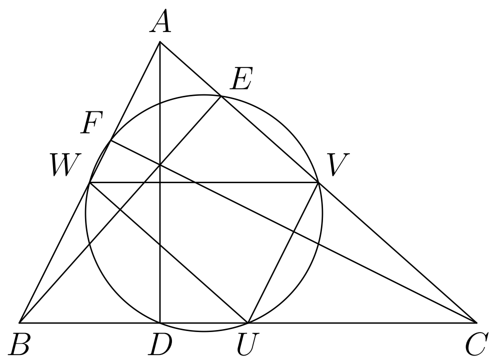
meaning $\cos B = 700/821$. Hence $\sin B = 429/821$. Applying the law of sines to $\triangle ABC$ gives

$$\frac{AC}{\sin B} = 2R = 821.$$

Finally, $AC = 429$. □

Now we give the proof I promised of the existence of the nine-point circle.

Theorem 2.2 (9-pt Circle). *In $\triangle ABC$, U, V, W are the midpoints of segments BC, CA, AB. Let D, E, F be the feet of the $A-$, $B-$, $C-$ altitudes respectively. Denote by K, L, M the midpoints of HA, HB, HC where H is the orthocenter of $\triangle ABC$. Prove U, V, W, D, E, F, K, L, M are concyclic.*



Solution. The first 6 points were due to Brianchon in 1821. The points K, L, and M were added by Terquem (see [4]). First, draw in the circumcircle of triangle UVW. By using power of a point and radical axis we will show that the other points lie on this circle as well.

To prove $UDWF$ is cyclic it suffices to show $(BD)(BU) = (BW)(BF)$. Note that $BD = c\cos B$ and $BU = a/2$. Furthermore, $BW = a\cos B$ and $BF = c/2$. Thus, $(BD)(BU) = (BW)(BF)$, establishing that $UDWF$ is cyclic.

By symmetry, $WFEV$ and $EVUD$ are also cyclic. Assume by way of contradiction that the circumcircles of $UDWF$, $WFEV$ and $EVUD$ do not coincide. By the radical axis theorem on those three circles, WF, EV and UD must concur. However this is impossible! Thus, the circumcircles coincide, and we have established that $WFEVUD$ is cyclic.

It remains to prove that K, L, M lie on this circle. By power of a point it suffices to show

$$(AK)(AD) = (AF)(AW).$$

Plugging $AK = R\cos A$, and $AD = bc/2R$ gives

$$(AK)(AD) = bc\cos A/2 = (c/2)b\cos A = (AF)(AW),$$

as needed. By symmetry, L and M lie on the circle as well. □

The good news is we do not have to repeat this for an obtuse triangle because we have used the notion of directed distances. An experienced reader would have noted that points K, L, M lie on the nine-point circle trivially by the orthocentric tetrad. Indeed, after we proved that the outer 6 points were concyclic, we could apply that result to triangles HAB and HBC to show that the midpoints of the segments joining the orthocenter to the vertices also lay on the circle.

Example 2.2 (Turkey 1999). In acute triangle ABC with circumradius R, altitudes AD, BE, CF have lengths h_1, h_2 and h_3 respectively. If t_1, t_2 and t_3 are the lengths of the tangents from A, B and C to the circumcircle of triangle DEF, prove that

$$\sum_{i=1}^{3}\left(\frac{t_i}{\sqrt{h_i}}\right)^2 \leq \frac{3}{2}R.$$

Solution. Write $\sum_{i=1}^{3}\left(\frac{t_i}{\sqrt{h_i}}\right)^2$ as $\sum_{i=1}^{3}\left(\frac{t_i^2}{h_i}\right)$. Actually, does the t_i^2 term look familiar? It's just the power of points A, B and C with respect to the nine point circle. Specifically, we have $t_1^2 = bc\cos A/2$, and similar formulas for the others. What about h_i? Well, $h_1 = bc/2R$ from the remark in Section 2, so we have

$$\sum_{i=1}^{3}\left(\frac{t_i}{\sqrt{h_i}}\right)^2 = R\cos A + R\cos B + R\cos C.$$

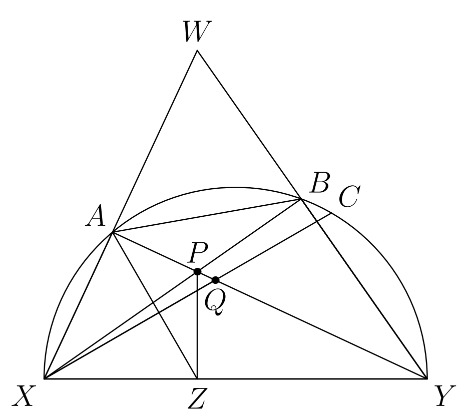
Thus, it suffices to show $\cos A + \cos B + \cos C \leq \frac{3}{2}$. But this follows by Jensen's inequality, and we are done. □

3 A Non-Trivial Example

Our last example is from the USAJMO in 2013. It illustrates the use of the orthocentric distances with law of sines.

Example 3.1 (USAJMO 2013) Quadrilateral $XABY$ is inscribed in semicircle ω with diameter XY. Let $P = \overline{AY} \cap \overline{BX}$. Point Z is the foot of the perpendicular from P to XY. Point C is on ω such that XC is perpendicular to AZ. Let $Q = \overline{AY} \cap \overline{XC}$. Prove that

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{AY}{AX}.$$



Solution. It does not look as if we have an orthocenter to apply our distances to. Or do we? Letting $W = \overline{AX} \cap \overline{BY}$, we see YA is perpendicular to WX and XB is perpendicular to WY. Since $P = \overline{AY} \cap \overline{BX}$, it is the orthocenter of triangle WXY. For brevity, I will write $\cos X$ for $\cos\angle WXY$, and $\cos Y$ for $\cos\angle WYX$.

Let $XY = w$, $WX = y$, and $YW = x$. Then,

$$\frac{AY}{AX} = \frac{wx/2R}{w\cos X} = \frac{x}{2R\cos X}$$

and

$$\frac{BY}{XP} = \frac{w\cos Y}{2R\cos X}.$$

Now we just need CY/XQ. Applying the law of sines to $\triangle CXY$ gives

$$\frac{CY}{\sin\angle CXY} = \frac{XY}{\sin 90^\circ} = XY = w.$$

Actually because $\triangle XAZ \sim \triangle XYW$, we have $\angle CXY = \angle WXB = 90 - \angle W$. Thus $CY = w\cos W$. Now we calculate XQ. The law of sines on $\triangle QAX$ gives

$$\frac{XQ}{\sin 90^\circ} = \frac{w\cos X}{\sin\angle AQX}.$$

Remark that $\angle AQX = 90^\circ - \angle AXQ = 90 - \angle YXB = \angle Y$, where the second-to-last step follows from $\triangle XAZ \sim \triangle XYW$. Hence, $XQ = w\cos X/\sin Y$. Therefore,

$$\begin{aligned}\frac{BY}{XP} + \frac{CY}{XQ} &= \frac{w\cos Y}{2R\cos X} + \frac{\cos W\sin Y}{\cos X}\\ &= \frac{w\cos Y}{2R\cos X} + \frac{y\cos W}{2R\cos X}\\ &= \frac{x}{2R\cos X}\\ &= \frac{AY}{AX},\end{aligned}$$

as wanted. □

4 Practice Problems

We end this article with several practice problems.

Problem 4.1 (TKMT, David Altizio) Let ABC be a triangle with $AB = 3$ and $AC = 4$. Points O and H denote the circumcenter and orthocenter of $\triangle ABC$ respectively. If $OH || BC$, what is $\cos A$?

Problem 4.2 (HMMT November 2016) Let ABC be a triangle with $AB = 5$, $BC = 6$, and $AC = 7$. Let its orthocenter be H and the feet of the altitudes from A, B, C to the opposite sides be D, E, F respectively. Let DF intersect the circumcircle of AHF again at X. Find the length of EX.

Problem 4.3 (HMMT November 2010) Triangle ABC is given with $AB = 13, BC = 14, CA = 15$. Let E and F be the feet of the altitudes from B and C respectively. Let G be the foot of the altitude from A in triangle AFE. Find AG.

Problem 4.4 (APMO 2013) Let ABC be an acute triangle with altitudes $\overline{AD}, \overline{BE}, \overline{CF}$, and let O be the circumcenter of $\triangle ABC$. Show segments OA, OF, OB, OD, OC, and OE dissect triangle ABC into three pairs of triangles with equal area.

Problem 4.5 (APMO 2004) Let O and H be the circumcenter and orthocenter of an acute $\triangle ABC$ respectively. Prove that the area of one of triangles AOH, BOH, and COH is equal to the sum of the areas of the other two.

Problem 4.6 (IMO 2008) Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2. Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.

Problem 4.7 (Tuymaada 2002) The points D and E on the circumcircle of an acute triangle ABC are such that $AD = AE = BC$. Let H be the common point of the altitudes of triangle ABC. Given that $AH^2 = BH^2 + CH^2$, prove that H lies on the segment DE.

Problem 4.8 (Tuymaada 2010) Let ABC be an acute triangle, H its orthocenter, D a point on the side $\overline{BC}$, and P a point such that $ADPH$ is a parallelogram. Show that $\angle BPC > \angle BAC$.

Finally, I will finish with a few nice inequalities.

Problem 4.9 (Britain 2011) Let triangle ABC be acute. The feet of the altitudes from A, B and C are D, E and F respectively. Prove that $DE + DF \leq BC$ and determine when equality holds.

Problem 4.10 (Canada 2015) Let triangle ABC have altitudes AD, BE and CF. Denote by H the orthocenter of triangle ABC. Prove that

$$\frac{AB \cdot AC + BC \cdot CA + CA \cdot CB}{AH \cdot AD + BH \cdot BE + CH \cdot CF} \leq 2.$$

Problem 4.11 In $\triangle ABC$ let D, E and F be the feet of the altitudes from A, B and C, respectively. Let K be the intersection of AO with BC, where O is the circumcenter of triangle ABC. Prove that

$$\frac{DE}{DF} = \frac{KB}{KC}.$$

Some hints and answers to the problems are presented in the next section.

5 Hints

Problem 4.1 If $OH||BC$, the distance from H and O to BC must be the same. The answer is

$$\frac{25-\sqrt{113}}{32}.$$

Problem 4.2 First things first: Power of a point. Prove $DX = DE$. Why does DA bisect $\angle XDE$? DA is the perpendicular bisector of XE. The answer is $190/49$.

Problem 4.3 Similarity: $\triangle AFE \sim \triangle ACB$. Add in the foot of the altitude from A to BC. The answer is $396/65$.

Problem 4.4 Prove $[AOF] = [COD]$. Invoke symmetry.

Problem 4.5 Sine areas:

$$[AOH] = \frac{(AO)(AH)}{2}\sin\angle HAO.$$

Problem 4.6 Prove $(AB_1)(AB_2) = (AC_1)(AC_2)$. To do this let N be the midpoint of AC. Find HN by using Stewart's.

Problem 4.7 Draw a circle at A with radius BC. Consider radical axis. Use Corollary 2.3.

Problem 4.8 You need $\cot\angle BPC < \cot\angle BAC$. Use the cotangent angle sum formula on the appropriate triangles.

Problem 4.9 The condition is $AB = AC$. Expand the $\cos B$ and $\cos C$, and bash away.

Problem 4.10 Again, bash away. You should be left with

$$ab + bc + ca \le a^2 + b^2 + c^2.$$

Problem 4.11 You will need to use the fact that AO and AH are isogonal. The exact result is

$$\frac{BD}{DC}\cdot\frac{BK}{KC} = \frac{c^2}{b^2}.$$

Acknowledgements

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6 References

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