SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2018: 44(2), p. 69–72.

4300$\star$. *Proposed by Leonard Giugiuc.*

Let a, b and c be positive real numbers with $a+b+c = ab+bc+ca > 0$. Prove or disprove that

$$\sqrt{24ab+25}+\sqrt{24bc+25}+\sqrt{24ca+25} \geq 21.$$

Arkady Alt pointed out some defects in the solution of 4300 published in 43(10). We present a complete solution here.

We received 3 submissions of which 2 were correct and complete. We feature the solution by the proposer modified by the editor.

Let $a+b+c = ab+bc+ac = k$. Since $(a+b+c)^2 \geq 3(ab+bc+ac)$, we have $k^2 \geq 3k$, and since $k > 0$ it follows that $k \geq 3$. Assume, without loss of generality, that $bc = \max\{ab, bc, ac\}$. Then $bc \geq 1$, and $a = \frac{b+c-bc}{b+c-1}$. Setting $b+c = 2s$ and $bc = p^2$, we know that $s \geq p$, $p \geq 1$, and $a = \frac{2s-p^2}{2s-1}$. We consider two cases.

Case 1: $1 \leq p \leq 2$. For any such p, let $f_p(t) = \frac{2s-p^2}{2t-1}$ on $[p, \infty)$. Since $p^2 \geq 1$, we know that f_p is increasing, which implies that $a = \frac{2s-p^2}{2s-1} \geq \frac{2p-p^2}{2p-1}$. If we apply the AM-GM inequality we have:

$$\sqrt{24ab+25}+\sqrt{24ac+25} \geq 2((24ab+25)(24ac+25))^{1/4}.$$

Then

$$\begin{aligned}
&2((24ab+25)(24ac+25))^{1/4}\\
&= 2\left(576\left(\frac{2s-p^2}{2s-1}\right)^2 p^2 + 1200\left(\frac{2s-p^2}{2s-1}\right)s + 625\right)^{1/4}\\
&\geq 2\left(576\left(\frac{2p-p^2}{2p-1}\right)^2 p^2 + 1200\left(\frac{2p-p^2}{2p-1}\right)p + 625\right)^{1/4}\\
&= 2\left(\left[24\left(\frac{2-p}{2p-1}\right)p^2+25\right]^2\right)^{1/4}\\
&= 2\sqrt{24\left(\frac{2-p}{2p-1}\right)p^2+25}.
\end{aligned}$$

So we have

$$\sqrt{24ab+25}+\sqrt{24ac+25}\geq 2\sqrt{24\left(\frac{2-p}{2p-1}\right)p^2+25},$$

and $\sqrt{24bc+25}=\sqrt{24p^2+25}$. So it suffices to show that

$$\sqrt{24p^2+25}+2\sqrt{24\left(\frac{2-p}{2p-1}\right)p^2+25}\geq 21.$$

Since $1\leq p\leq 2$, it follows that

$$\sqrt{24p^2+25}+2\sqrt{24\left(\frac{2-p}{2p-1}\right)p^2+25}\geq\sqrt{24p^2+25}+2\sqrt{73-24p}$$

Using basic calculus, it follows that the function $f:[1,2]\longrightarrow\mathbb{R}$,

$$f(p)=\sqrt{24p^2+25}+2\sqrt{73-24p}$$

is strictly decreasing on $[1,5/3]$ and it is strictly increasing on $[5/3,2]$. Hence it has a minimum at $5/3$ and the minimum value is more than 21.

Case 2: $p>2$. Then we have

$$\sqrt{24ab+25}+\sqrt{24ac+25}+\sqrt{24bc+25}\geq 10+\sqrt{24p^2+25}>10+11=21.$$

4311. *Proposed by Mihaela Berindeanu.*

Let A and B be two matrices in $\mathfrak{M}_3(\mathbb{Z})$ with $AB=BA$ and $\det A=\det B=1$. Find the possible values for $\det\left(A^2+B^2\right)$ knowing that

$$\det\left(A^2+2AB+4B^2\right)-\det\left(A^2-2AB+4B^2\right)=-4.$$

We received 4 correct solutions and one incomplete submission. We will feature a solution by Leonard Guigiuc.

Since $\det(B)=1$, then $\det(B^{-1})=1$ and $B^{-1}\in M_3(\mathbb{Z})$. This implies that $\det(AB^{-1})=1$ and $AB^{-1}\in M_3(\mathbb{Z})$. Since $AB=BA$, we set $C=AB^{-1}$ to get

$$\det(A^2+2AB+4B^2)-\det(A^2-2AB+4B^2)=\det(C^2+2C+4I_3)-\det(C^2-2C+4I_3),$$

and $\det(A^2+B^2)=\det(C^2+I_3)$.

Consider the polynomial $f(x)=\det(C-xI_3)$ over $\mathbb{C}$. Then $f(x)=1-kx+tx^2-x^3$ for all $x\in\mathbb{C}$ with k and t integers.

Let $u=1/2(-1+i\sqrt{3})$. Then $C^2+2C+4I_3=(C-2uI_3)(C-2u^2I_3)$, which implies

$$\begin{aligned}\det(C^2+2C+4I_3) &= \det(C-2uI_3)\cdot\det(C-2u^2I_3)\\ &= f(2u)\cdot f(2u^2)\\ &= 49+4k^2+16t^2-14k+28t+8kt.\end{aligned}$$

Similarly,

$$\det(C^2 - 2C + 4I_3) = 81 + 4k^2 + 16t^2 - 18k - 36t - 8kt.$$

Thus

$$\det(C^2 + 2C + 4I_3) - \det(C^2 - 2C + 4I_3) = -4$$

if and only if $k + 16t + 4kt = 7$ or, equivalently, $4t(k + 4) = 7 - k$. Since clearly, $k \neq -4$, we have $4t = -1 + 11/(k + 4)$, which implies $11/(k + 4) \in \mathbb{Z}$. Hence $k + 4 \in \{\pm 1, \pm 11\}$. Since t is an integer as well, we obtain $k = -5$ and $t = 3$ or $k = 7$ and $t = 0$. Observe that

$$\det(C^2 + I_3) = \det(C - I_3) \cdot \det(C + I_3) = f(i) \cdot f(-i) = (k - 1)^2 + (t - 1)^2.$$

In conclusion, $\det(A^2 + B^2) \in \{37, 52\}$.

4312. *Proposed by William Bell.*

Prove that

$$\sum_{r=1}^{\infty} \frac{1}{2^r} \tanh\left(\frac{x}{2^r}\right) = \coth x - \frac{1}{x}.$$

Nine correct solutions were received. Six followed the strategy of the first solution and three the strategy of the second.

Solution 1.

Since $\tanh u = 2 \coth 2u - \coth u$,

$$\sum_{r=1}^{n} \frac{1}{2^r} \tanh \frac{x}{2^r} = \sum_{r=1}^{n} \left(\frac{1}{2^{r-1}} \coth \frac{x}{2^{r-1}} - \frac{1}{2^r} \coth \frac{x}{2^r} \right) = \coth x - \frac{1}{2^n} \coth \frac{x}{2^n}.$$

Since, by l'Hôpital's Rule, for example, $\lim_{v \to 0} v \coth v = 1$,

$$\sum_{r=1}^{\infty} \frac{1}{2^r} \tanh \frac{x}{2^r} = \coth x - \frac{1}{x} \lim_{n \to \infty} \frac{x}{2^n} \coth \frac{x}{2^n} = \coth x - \frac{1}{x}.$$

Solution 2.

For $n \geq 1$, consider the product

$$P_n(x) = \cosh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{4}\right) \ldots \cosh\left(\frac{x}{2^n}\right).$$

Since

$$P_n(x) \sinh\left(\frac{x}{2^n}\right) = \frac{1}{2} P_{n-1}(x) \sinh\left(\frac{x}{2^{n-1}}\right),$$

an induction argument leads to

$$P_n(x) = \frac{\sinh x}{2^n \sinh\left(\frac{x}{2^n}\right)}.$$

With D denoting differentiation, we have that

$$\begin{aligned}\sum_{r=1}^{n} \frac{1}{2^r} \tanh \frac{x}{2^r} &= D\left(\sum_{r=1}^{n} \ln \cosh \frac{x}{2^r}\right) = D \ln \left(\prod_{r=1}^{n} \cosh \frac{x}{2^r}\right) \\ &= D \ln P_n(x) - D(\ln \sinh x - n \ln 2 - \ln \sinh \frac{x}{2^n}) \\ &= \coth x - \frac{1}{2^n} \coth \frac{x}{2^n},\end{aligned}$$

Let $n \to \infty$ to obtain

$$\sum_{r=1}^{\infty} \frac{1}{2^r} \tanh \left(\frac{x}{2^r}\right) = \coth x - \frac{1}{x}.$$

4313. *Proposed by Marian Cucoanes and Leonard Giugiuc.*

Let I be the incenter of triangle ABC, and denote by H_a, H_b and H_c the orthocenters of triangles IBC, ICA and IAB, respectively. Prove that triangles ABC and $H_aH_bH_c$ have the same area.

We received six submissions, all correct, and feature the solution by Mohammed Aassila.

The result holds for any point P in the plane of triangle ABC that is not on a line joining two of its vertices; the argument is therefore not restricted to the special case of the problem, namely $P = I$. It is based on a familiar property of the mixed product $[\overrightarrow{p}, \overrightarrow{q}]$, which for vectors $\overrightarrow{p} = \langle p_1, p_2 \rangle$ and $\overrightarrow{q} = \langle q_1, q_2 \rangle$ in $\mathbf{R}^2$ is just the determinant,

$$[\overrightarrow{p}, \overrightarrow{q}] = \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = p_1q_2 - p_2q_1.$$

Specifically, given a pair of triangles ABC and $A'B'C'$ in the plane, we have

$$[\overrightarrow{BA'}, \overrightarrow{AB'}] + [\overrightarrow{CB'}, \overrightarrow{BC'}] + [\overrightarrow{AC'}, \overrightarrow{CA'}] = [\overrightarrow{A'B'}, \overrightarrow{A'C'}] - [\overrightarrow{AB}, \overrightarrow{AC}].$$

This identity can be found as a straightforward exercise in texts that deal with the mixed product. Of course, when A', B', and C' are, respectively, the orthocentres of triangles PBC, PCA, and PAB, then each of the quantities on the left are zero (because the vectors $\overrightarrow{BA'}$ and $\overrightarrow{AB'}$ are both perpendicular to the line PC, etc.). We are therefore left with

$$[\overrightarrow{A'B'}, \overrightarrow{A'C'}] = [\overrightarrow{AB}, \overrightarrow{AC}],$$

which says (under the assumption that A', B', C' are the appropriate orthocentres) that the areas of triangles ABC and $A'B'C'$ are equal.

4314. *Proposed by Michel Bataille.*

Let n be a positive integer. Evaluate in closed form

$$\sum_{k=1}^{n} k2^k \cdot \frac{\binom{n}{k}}{\binom{2n-1}{k}}.$$

We received three solutions, and we present two of them.

Solution 1, by Paolo Perfetti, slightly edited.

Denote the k^{th} summand by s_k; that is,

$$s_k = k2^k \frac{\binom{n}{k}}{\binom{2n-1}{k}} = k2^k \frac{\binom{n}{k} k!(2n-k-1)!}{(2n-1)!}.$$

We will use the beta function, $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ defined for complex x, y such that $\mathfrak{Re}(x), \mathfrak{Re}(y) > 0$. One of the well known properties of the beta function (which follows from its relationship to the gamma function) is that $\beta(x,y) = \frac{(x-1)!(y-1)!}{(x+y-1)!}$ for x, y positive integers. Hence we can write

$$s_k = k \cdot 2^k \cdot \binom{n}{k} \cdot 2n \cdot \beta(k+1, 2n-k),$$

and we have

$$\begin{aligned}\sum_{k=1}^{n} s_k &= \sum_{k=1}^{n} \left(k2^k \binom{n}{k} \cdot 2n \int_0^1 t^k (1-t)^{2n-k-1}\,dt \right) \\ &= 2n \int_0^1 (1-t)^{2n-1} \sum_{k=1}^{n} k \binom{n}{k} \left(\frac{2t}{1-t} \right)^k dt. \qquad (1)\end{aligned}$$

Note that for any y we have the equality $\sum\limits_{k=1}^{n} k\binom{n}{k} y^k = ny(y+1)^{n-1}$:

$$\sum_{k=1}^{n} k \binom{n}{k} y^k = n \sum_{k=1}^{n} \binom{n-1}{k-1} y^k = ny \sum_{k=0}^{n-1} \binom{n-1}{k} y^k = ny(1+y)^{n-1},$$

with the last equality following from the binomial theorem. Thus, continuing from (1) we get

$$\begin{aligned}\sum_{k=1}^{n} s_k &= 2n \int_0^1 (1-t)^{2n-1} \left(\frac{2nt}{1-t} \right) \left(\frac{2t}{1-t} + 1 \right)^{n-1} dt \\ &= 2n^2 \int_0^1 (1-t)^{n-1} (2t)(1+t)^{n-1}\,dt \\ &= 2n^2 \int_0^1 2t(1-t^2)^{n-1}\,dt \\ &= 2n^2 \cdot \left(-\frac{1}{n} \right) \cdot (1-t^2)^n \Big|_0^1 = 2n.\end{aligned}$$

Solution 2, by the proposer.

Denote the k^{th} summand by s_k as in the previous solution.

Let

$$u_k = -2^k \frac{\binom{n}{k}}{\binom{2n}{k}} = -2^k \frac{n!(2n-k)!}{(2n)!(n-k)!}$$

for $k = 1, 2, \ldots, n$ and $u_{n+1} = 0$. Note that for $k = 1, 2, \ldots, n-1$ we have

$$\begin{aligned} u_{k+1} - u_k &= 2^k \cdot \frac{n!}{(2n)!} \cdot \frac{(2n-k-1)!}{(n-k-1)!} \cdot \left(-2 + \frac{2n-k}{n-k}\right) \\ &= 2^k \cdot \frac{n!}{(2n)!} \cdot \frac{(2n-k-1)!}{(n-k-1)!} \cdot \frac{k}{n-k} \\ &= k \cdot 2^k \cdot \frac{n!}{(2n)!} \cdot \frac{(2n-k-1)!}{(n-k)!} = \frac{s_k}{2n}. \end{aligned}$$

The equality $u_{k+1} - u_k = \frac{s_k}{2n}$ holds trivially for $k = n$ since $u_{n+1} = 0$.

It follows that

$$\sum_{k=1}^{n} s_k = 2n \left(\sum_{k=1}^{n} u_{k+1} - u_k \right) = 2n(u_{n+1} - u_1) = 2n.$$

4315. *Proposed by Moshe Stupel, modified by the editors.*

Let H be the orthocenter of triangle ABC, and denote by R, r, and r' respectively the circumradius, inradius, and radius of the excircle that is opposite vertex A. Prove that $HA + r' = 2R + r$.

We received 11 submissions, all substantially correct. We present a combination of the solutions from Leonard Giugiuc and Cristóbal Sánchez-Rubio.

The statement of the problem is not quite correct. The identity we shall prove is

$$2R\cos A + r' = 2R + r. \tag{1}$$

Editor's comments. Giugiuc and just one other solver (Pranesachar) stated explicitly that $HA = 2R\cos A$ when $0° < \angle A \le 90°$; otherwise (when $\angle A$ is obtuse) $HA = -2R\cos A$. All other submissions tacitly took HA to be a directed length by setting it equal to $2R\cos A$. If you prefer to assume lengths to be nonnegative, then the correct identity in terms of HA is $r' \pm HA = 2R + r$, with the negative sign used when $\angle A$ is obtuse.

We shall make use of familiar formulas for the circumradius

$$R = \frac{abc}{4sr}, \tag{2}$$

and for the area,

$$rs = r'(s-a) = \sqrt{s(s-a)(s-b)(s-c)}. \tag{3}$$

Using expressions for R and r' (from (2) and (3)) reduces identity (1) (in the form $2R(\cos A - 1) = r - r'$) to

$$2\frac{abc}{4sr}(\cos A - 1) = r\left(1 - \frac{s}{s-a}\right) = -\frac{ra}{s-a},$$

so that the identity to be established is equivalent to

$$\frac{bc\,(1-\cos A)}{2sr} = \frac{r}{s-a}. \tag{4}$$

But because

$$\begin{aligned} 1-\cos A &= 1 + \frac{a^2-b^2-c^2}{2bc} \\ &= \frac{a^2-(b-c)^2}{2bc} \\ &= \frac{(a+b-c)\,(a-b+c)}{2bc} \\ &= \frac{2\,(s-c)\cdot 2\,(s-b)}{2bc} \\ &= \frac{2\,(s-b)\,(s-c)}{bc}, \end{aligned}$$

equation (4) reduces to

$$(s-a)\,(s-b)\,(s-c) = sr^2,$$

which is the square of Heron's formula (see (3) above).

4316. *Proposed by Daniel Sitaru.*

Let $f : [0, 11] \to \mathbb{R}$ be an integrable and convex function. Prove that

$$\int_3^5 f(x)dx + \int_6^8 f(x)dx \le \int_0^2 f(x)dx + \int_9^{11} f(x)dx.$$

Ten correct solutions were received. Most followed the procedure of Solution 2.

Solution 1, by Roy Barbara.

Let $g(x) = ax + b$ be the linear function that satisfies $g(3) = f(3)$ and $g(8) = f(8)$.

Because $f(x)$ is convex, $f(x) \ge g(x)$ when $0 \le x \le 3$ or $8 \le x \le 10$, and $f(x) \le g(x)$ when $3 \le x \le 8$. The left side does not exceed

$$\int_3^5 g(x)\,dx + \int_6^8 g(x)\,dx = 22a + 4b,$$

and the right side is not less than

$$\int_0^2 g(x)\,dx + \int_9^{11} g(x)\,dx = 22a + 4b.$$

The result follows.

Solution 2.

Since $f(x)$ is convex,

$$f(3+x) \le \frac{2}{3}f(x) + \frac{1}{3}f(9+x) \quad \text{and} \quad f(6+x) \le \frac{1}{3}f(x) + \frac{2}{3}f(9+x).$$

Therefore,

$$\begin{aligned}\int_3^5 f(x)\,dx + \int_6^8 f(x)\,dx &= \int_0^2 [f(3+x) + f(6+x)]\,dx \\ &\le \int_0^2 [f(x) + f(9+x)]\,dx \\ &= \int_0^2 f(x)\,dx + \int_9^{11} f(x)dx.\end{aligned}$$

Solution 3, by Oliver Geupel.

Recall the Hermite-Hadamard Inequality for convex functions:

$$(b-a)f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)\,dx \le \frac{1}{2}(b-a)(f(a)+f(b)).$$

Therefore

$$\begin{aligned}\int_3^5 f(x)\ dx &\le f(3) + f(5) \\ &\le \left(\frac{7}{9}f(1) + \frac{2}{9}f(10)\right) + \left(\frac{5}{9}f(1) + \frac{4}{9}f(10)\right) \\ &= \frac{4}{3}f(1) + \frac{2}{3}f(10),\end{aligned}$$

and, similarly,

$$\int_6^8 f(x)\,dx \le f(6) + f(8) \le \frac{2}{3}f(1) + \frac{4}{3}f(10).$$

Therefore

$$\int_3^5 f(x)\,dx + \int_6^8 f(x)\,dx \le 2f(1) + 2f(10) \le \int_0^2 f(x)\,dx + \int_9^{11} f(x)\,dx.$$

4317. *Proposed by Leonard Giugiuc.*

Solve the following system of equations over reals:

$$\begin{cases} a+b+c+d=4, \\ abc+abd+acd+bcd=2, \\ abcd=-\frac{1}{4}. \end{cases}$$

We received nine correct submissions. We present the solution by Oliver Geupel, modified and expanded by the editor.

Let $p=a+b, q=ab, r=c+d, s=cd$. Then $q\neq 0$ and $s=-\dfrac{1}{4q}$.

Since

$$2=ab(c+d)+cd(a+b)=qr+ps=q(4-p)-\frac{p}{4q},$$

we have

$$8q=4q^2(4-p)-p=16q^2-(4q^2+1)p$$

so

$$p=\frac{8q(2q-1)}{4q^2+1} \qquad \text{and} \qquad r=4-p=\frac{4(2q+1)}{4q^2+1}.$$

Since the quadratic x^2-px+q has two real roots a and b, its discriminant is non-negative.

By labourious computations, we find that

$$p^2-4q=\frac{64q^2(2q-1)^2}{(4q^2+1)^2}-4q=\frac{4q}{(4q^2+1)^2}D$$

where

$$\begin{aligned} D &= 16q(4q^2-4q+1)-(16q^4+8q^2+1) \\ &= -16q^4+64q^3-72q^2+16q-1 \\ &= -16(q^4-4q^3+\tfrac{9}{2}q^2-q+\tfrac{1}{16}) \\ &= -16(q^2-2q+\tfrac{1}{4})^2 \\ &= -16((q-1+\tfrac{\sqrt{3}}{2})(q-1-\tfrac{\sqrt{3}}{2}))^2. \end{aligned}$$

Since $p^2-4q^2\geq 0$ and $q\neq 0$, it follows that

$$\text{either} \quad q<0 \quad \text{or} \quad q=1\pm\frac{\sqrt{3}}{2} \tag{1}$$

Similarly, since the quadratic x^2-rx+s has two real roots c and d, its discriminant is non-negative.

By labourious computations, we find that

$$\begin{aligned} r^2 - 4s^2 &= \frac{16(2q+1)^2}{(4q^2+1)^2} + \frac{1}{q} \\ &= \frac{16q(2q+1)^2 + (4q^2+1)^2}{q(4q^2+1)^2} \\ &= \frac{16}{q} \cdot \frac{q^4 + 4q^3 + \frac{9}{2}q^2 + q + \frac{1}{16}}{(4q^2+1)^2} \\ &= \frac{16}{q}\left(\frac{q^2 + 2q + \frac{1}{4}}{4q^2+1}\right)^2 \\ &= \frac{16}{q}\left(\frac{(q+1+\frac{\sqrt{3}}{2})(q+1-\frac{\sqrt{3}}{2})}{4q^2+1}\right)^2. \end{aligned}$$

Since $r^2 - 4s^2 \geq 0$ it follows that

$$\text{either} \quad q > 0 \quad \text{or} \quad q = -1 \pm \frac{\sqrt{3}}{2}. \tag{2}$$

From (1) and (2) it is readily seen that there are four possible values of q given by $q = \pm 1 \pm \frac{\sqrt{3}}{2}$.

For the first case $q = 1 + \frac{\sqrt{3}}{2}$, we have

$$\begin{aligned} p &= \frac{8q(2q-1)}{4q^2+1} = 1 + \sqrt{3}, \\ r &= 4 - p = 3 - \sqrt{3}, \\ s &= \frac{-1}{4q} = \frac{-1}{4+2\sqrt{3}} = -1 + \frac{\sqrt{3}}{2}. \end{aligned}$$

Solving the equations

$$a + b = p,\ ab = q,\ c + d = r,\ cd = s,$$

we find by tedious calculations that

$$(a, b, c, d) = \left(\frac{1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}, \frac{5-3\sqrt{3}}{2}\right). \tag{3}$$

Similarly, the second case $q = 1 - \frac{\sqrt{3}}{2}$ yields

$$p = 1 - \sqrt{3},\ r = 3 + \sqrt{3},\ s = 1 - \tfrac{\sqrt{3}}{2},$$

which would lead to the solution

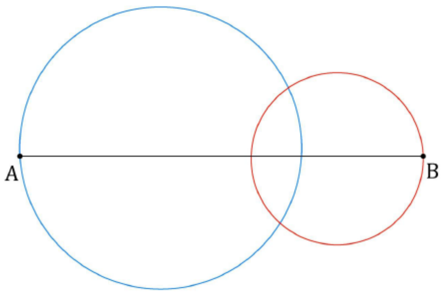
$$(a, b, c, d) = \left(\frac{1-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, \frac{5+3\sqrt{3}}{2}\right). \tag{4}$$

The third case $q = -1 + \frac{\sqrt{3}}{2}$ is symmetric to the first case with p and q interchanged with r and s, respectively, leading to a permutation of the solution in (3). Finally, the fourth case $q = -1 - \frac{\sqrt{3}}{3}$ would eventually lead to a permutation of the solution in (4).

In conclusion, all the solutions are given by the two ordered quadruples together with all of their permutations.

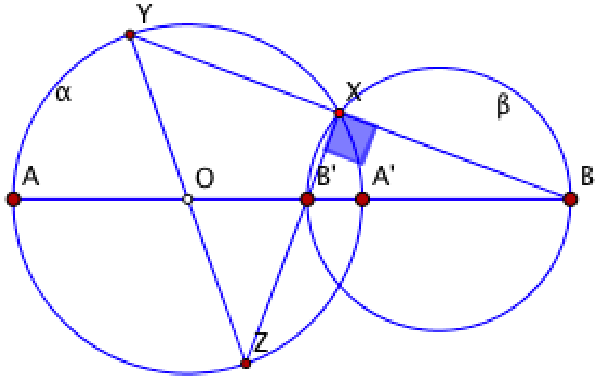
4318. *Proposed by Thanos Kalogerakis.*

Given a pair of intersecting circles (just their circumferences, not their centres), let AB be the common diameter with one end on each circle and neither end inside either circle. Show how to construct the midpoint of AB using only a straightedge and prove that your construction is correct.



We received four submissions, but one was incomplete. We present the solution sent in by the Missouri State University Problem-Solving Group, modified by the editor.

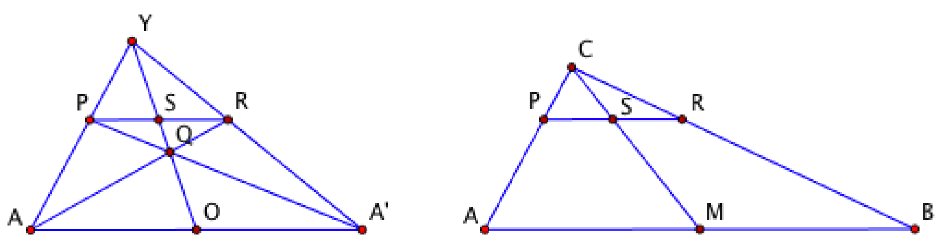
Denote the circle containing A by α, the circle containing B by β, one of the points where α and β intersect by X, and the point other than B where AB meets β by B'. Our preliminary step is to construct the center of α. We note that once we have the center of α, by the Poncelet-Steiner Theorem we can construct any point constructible with compass and straightedge using straightedge alone.



Since $\angle B'XB$ is inscribed in a semi-circle, XB' is perpendicular to XB. If XB is not tangent to α, it meets α in another point Y. Let Z denote the second point where XB' meets α. Since $\angle YXZ$ is a right angle, YZ is a diameter of α and the intersection of YZ and AB gives the center of α, which we denote by O. If XB is tangent to α, then XB' contains a diameter of α so that $B' = O$.

We now construct the midpoint M of AB given that O is the midpoint of the diameter AA'. Choose P to be a point between A and Y on the segment AY, and define

$$Q = PA' \cap YO, \quad R = QA \cap YA', \quad \text{and} \quad S = YO \cap PR,$$



as in the figure on the left. It is a known property of the quadrangle $AA'RP$ that PR is parallel to AA' and, therefore, that S is the midpoint of PR. [For a proof featuring harmonic conjugates see the *Crux* article *"Problem Solver's Toolkit No. 5: Harmonic Sets Part 2"*, 2013: 174-177, especially Ex. 3, page 176.]

Finally, define $C = BR \cap AP$ and $M = CS \cap AB$, as in the figure on the right. Because S is the midpoint of the segment PR, M must be the midpoint of the parallel segment AB, as required.

Comment. The problem as stated dealt with intersecting circles. We can still construct the center of α by straightedge when the given circles α and β are tangent (so that X, A', and B' coincide in a single point). Choose any point P different from A and X on α such that PB intersects β in a second point Q, and α in a second point S. Denote the second point where XQ meets α by R. Now QR is perpendicular to PB since $\angle XQB$ is inscribed in a semicircle. Similarly, QR is perpendicular to AR since $\angle ARX$ is inscribed in a semicircle. Therefore PB is parallel to AR. The convex quadrilateral with vertices A, P, S, R is an isosceles trapezoid. Let T be the intersection of PR and AS and let U be the intersection of AP and RS. Then TU contains a diameter of α, so the intersection of TU and AB gives the center of α.

Editor's comments. The featured solution requires drawing 10 lines, but the incomplete submission found M drawing only 5 lines. Unfortunately that solution came as a diagram with no accompanying justification. Perhaps the author just wanted to challenge his fellow readers, so we pass along that challenge as one of this month's problems, namely number 4414.

4319. *Proposed by Marius Drăgan.*

Let $x_1, x_2, \ldots, x_n \in (0, +\infty)$, $n \geq 2$, $\alpha \geq \frac{3}{2}$ such that $x_1^\alpha + x_2^\alpha + \cdots + x_n^\alpha = n$. Prove the following inequality:

$$\prod_{i=1}^{n}(1 + x_i + x_i^{\alpha+1}) \leq 3^n.$$

The original proposal contained a typo corrected here.

We received 1 correct solution, by the proposer, and we present it here.

Setting $a = \dfrac{1}{\alpha} \in \left[0, \dfrac{2}{3}\right]$, we consider the function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \ln(1 + x^a + x^{a+1}).$$

We have

$$f'(x) = \frac{ax^{a-1} + (a+1)x^a}{1 + x^a + x^{a+1}},$$
$$f''(x) = \frac{-(a+1)x^{2a} - 2ax^{2a-1} - ax^{2a-2} + (a^2 + a)x^{a-1} + (a^2 - a)x^{a+2}}{(1 + x^a + x^{a+1})^2}.$$

We show that $f''(x) < 0$, for all $x \in (0, \infty)$. It will be sufficient to prove that for all $x \in (0, \infty)$,

$$-(a+1)x^{a+2} - 2ax^{a+1} - ax^a + (a^2 + a)x + a^2 - a \leq 0$$

or

$$(a^2 + a)x < a - a^2 + ax^a + 2ax^{a+1} + (a+1)x^{a+2}.$$

For this it will be enough to prove that

$$(a^2 + a)x < a - a^2 + 2ax^{a+1}$$

or

$$x < \frac{1-a}{a+1} + \frac{2}{a+1}x^{a+1}.$$

From the power mean inequality, we obtain

$$\frac{a}{1+a} \cdot \frac{1-a}{a} + \frac{1}{1+a} \cdot 2x^{a+1} \geq \left(\frac{1-a}{a}\right)^{\frac{a}{a+1}} (2x^{a+1})^{\frac{1}{1+a}} = \left(\frac{1-a}{a}\right)^{\frac{a}{a+1}} 2^{\frac{1}{1+a}}x.$$

It will thus be sufficient to prove that $\left(\dfrac{a}{1-a}\right)^a \leq 2$. But since $a \in \left[0, \dfrac{2}{3}\right]$, we have $1 \leq \dfrac{2}{1-a} \leq 2$ or $\left(\dfrac{a}{1-a}\right)^a \leq 2^a$. But since $2^a < 2$, the claimed inequality holds. Thus f is a concave function.

From Jensen's inequality we have

$$\sum_{i=1}^{n} f(x_i^{\alpha}) \le nf\left(\frac{\sum\limits_{i=1}^{n} x_i^{\alpha}}{n}\right)$$

or $\sum\limits_{i=1}^{n} f(x_i^{\alpha}) \le nf(1)$ or $\ln \prod\limits_{i=1}^{n}(1 + x_i + x_i^{\alpha+1}) \le n \ln 3$.

4320. *Proposed by Abhay Chandra.*

For positive real numbers a, b, c, d, prove that

$$(a+b)(a+c)(a+d)(b+c)(b+d)(c+d) \ge 16\,(a+b+c+d)\,\sqrt[4]{a^5b^5c^5d^5}.$$

We received 8 submissions, all correct. We present the proof by Šefket Arslanagić.

We first establish the following lemma.

Lemma If a, b, c, d are positive reals, then

$$(a+b)(b+c)(c+d)(d+a) \ge (a+b+c+d)(abc+bcd+cda+dab).$$

Proof We have $(a+b)(b+c)(c+d)(d+a)$

$$\begin{aligned}
&= (ac+bd+ad+bc)(ac+bd+ab+cd)\\
&= (ac+bd)^2 + \sum_{cyc} a^2(bc+cd+db)\\
&\ge 4abcd + \sum_{cyc} a^2(bc+cd+db)\\
&= (a+b+c+d)(abc+bcd+cda+dab), \text{as claimed.}
\end{aligned}$$

By the lemma and the *AM-GM Inequality*, we have

$$\begin{aligned}
&(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)\\
&\ge (a+b+c+d)(abc+bcd+cda+dab)\cdot 2\sqrt{ac}\cdot 2\sqrt{bd}\\
&\ge (a+b+c+d)4\sqrt[4]{a^3b^3c^3d^3}\cdot 4\sqrt{abcd}\\
&= 16(a+b+c+d)\sqrt[4]{a^5b^5c^5d^5}, \text{completing the proof.}
\end{aligned}$$

Equality holds if and only if $a = b = c = d$.