

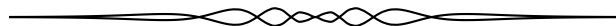
OLYMPIAD CORNER

No. 370

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **May 15, 2019**.



OC416. Given an acute nonisosceles triangle ABC with altitudes CD , AE , BF . Points E' and F' are symmetrical to E and F with respect to points A and B , respectively. Take a point C_1 on the ray \overrightarrow{CD} such that $DC_1 = 3CD$. Prove that $\angle E'C_1F' = \angle ACB$.

OC417. Point M is the midpoint of side BC of a triangle ABC in which $AB = AC$. Point D is the orthogonal projection of M onto side AB . Circle ω is inscribed in triangle ACD and tangent to segments AD and AC at K and L , respectively. Lines tangent to ω which pass through M intersect line KL at X and Y , where points X , K , L and Y lie on KL in this order. Prove that points M , D , X and Y are concyclic.

OC418. Three sequences (a_0, a_1, \dots, a_n) , (b_0, b_1, \dots, b_n) , $(c_0, c_1, \dots, c_{2n})$ of nonnegative real numbers are given such that for all $0 \leq i, j \leq n$ we have $a_i b_j \leq (c_{i+j})^2$. Prove that

$$\sum_{i=0}^n a_i \cdot \sum_{j=0}^n b_j \leq \left(\sum_{k=0}^{2n} c_k \right)^2.$$

OC419. Prove that there exist infinitely many positive integers m such that there exist m consecutive perfect squares with sum m^3 . Determine one solution with $m > 1$.

OC420. General Tilly and the Duke of Wallenstein play “Divide and rule!” (Divide et impera!). To this end, they arrange N tin soldiers in M companies and command them by turns. Both of them must give a command and execute it in their turn.

Only two commands are possible: The command “Divide!” chooses one company and divides it into two companies, where the commander is free to choose their size, the only condition being that both companies must contain at least one tin

soldier. On the other hand, the command “Rule!” removes exactly one tin soldier from each company.

The game is lost if in your turn you can't give a command without losing a company. Wallenstein starts to command.

- (a) Can he force Tilly to lose if they start with 7 companies of 7 tin soldiers each?
 (b) Who loses if they start with $M \geq 1$ companies consisting of $n_1 \geq 1, n_2 \geq 1, \dots, n_M \geq 1$ ($n_1 + n_2 + \dots + n_M = N$) tin soldiers?

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 15 mai 2019.

La rédaction souhaite remercier Valérie Lapointe, Carignan, QC, d'avoir traduit les problèmes.



OC416. Soit un triangle acutangle non isocèle ABC dont les hauteurs sont CD, AE et BF . E' et F' sont symétriques à E et F par rapport aux points A et B respectivement. On prend un point C_1 sur le segment CD tel que $DC_1 = 3CD$. Montrer que $\angle E'C_1F' = \angle ACB$.

OC417. Le point M est le milieu du côté BC d'un triangle ABC où $AB = AC$. Le point D correspond à la projection orthogonale de M sur le côté AB . Le cercle ω est inscrit dans le triangle ACD et est tangent aux segments AD et AC aux points K et L respectivement. Les droites tangentes à ω qui passent par M interceptent le segment KL en X et Y , où les points X, K, L et Y sont sur le segment KL dans cet ordre. Montrer que les points M, D, X et Y sont concycliques.

OC418. Soit trois suites $(a_0, a_1, \dots, a_n), (b_0, b_1, \dots, b_n), (c_0, c_1, \dots, c_{2n})$ de nombres réels non négatifs telles que pour toutes les suites $0 \leq i, j \leq n$ on a $a_i b_j \leq (c_{i+j})^2$. Montrer que

$$\sum_{i=0}^n a_i \cdot \sum_{j=0}^n b_j \leq \left(\sum_{k=0}^{2n} c_k \right)^2.$$

OC419. Montrer qu'il existe une infinité d'entiers positifs m tels qu'il existe m carrés parfaits consécutifs dont la somme est m^3 . Trouver une solution avec $m > 1$.

OC420. Le Général Tilly et le Duc de Wallenstein jouent à "diviser et régner!" Dans ce but, ils placent N soldats de plomb dans M groupes et leur donnent des commandes. Les deux doivent donner une commande et l'exécuter pendant leur tour. Seulement deux commandes sont possibles : la commande "Divise!" choisit un groupe et le divise en deux groupes où le commandant est libre de choisir la taille des groupes en autant que chaque groupe contienne au moins un soldat. La commande "Règne!" enlève exactement un soldat de plomb de chaque groupe. La partie est perdue si lors d'un tour, il n'est plus possible de donner un ordre sans perdre un groupe. C'est le Duc de Wallenstein qui débute les commandes.

- (a) Peut-il faire perdre le Général Tilly s'ils commencent avec 7 groupes de 7 soldats de plomb chacun ?
- (b) Qui perd s'ils commencent avec $M \geq 1$ groupes ayant $n_1 \geq 1, n_2 \geq 1, \dots, n_M \geq 1$ ($n_1 + n_2 + \dots + n_M = N$) soldats de plomb ?



OLYMPIAD CORNER

SOLUTIONS

Statements of the problems in this section originally appear in 2017: 43(10), p. 428–429 and 2018: 44(1), p. 13–14.

OC356. Suppose 2016 points of the circumference of a circle are colored red and the remaining points are colored blue. Given any natural number $n \geq 3$, prove that there is a regular n -sided polygon all of whose vertices are blue.

Originally Problem 5 of 2016 India National Olympiad.

We received 3 solutions. We present the solution by Ivo Dimitrić.

Without loss of generality, assume that the circle is unit and consider an arbitrary inscribed regular n -gon with at least one vertex red. Let $a > 0$ be the minimum arc-wise distance between distinct red points and let $b > 0$ the minimum arc-wise distance between blue vertices of the inscribed polygon (if any) and the red points. These minimum distances exist since the sets in question are finite. Let $\theta = \min\{a, b\}$. Then, rotate the selected n -gon through an angle $\theta/2$ (in radians). Then the vertices of the regular polygon would slide along the circumference the same arcwise distance equal to the angle of the rotation. Those vertices that were originally red will be rotated to blue points since $\theta/2 < a$, and none of the vertices that were originally blue can land at a red point under this rotation since $\theta/2 < b$, so they, too, will be rotated to blue points, and all the vertices of such rotated regular n -gon will be blue.

OC357. In $\triangle AEF$, let B and D be on segments AE and AF respectively, and let ED and FB intersect at C . Define K, L, M, N on segments AB, BC, CD, DA such that $\frac{AK}{KB} = \frac{AD}{BC}$ and its cyclic equivalents. Let the incircle of $\triangle AEF$ touch AE, AF at S, T respectively; let the incircle of $\triangle CEF$ touch CE, CF at U, V respectively. Prove that K, L, M, N concyclic implies S, T, U, V concyclic.

Originally Problem 2, Day 1 of 2016 China National Olympiad.

We received no solutions to this problem.

OC358. Prove that if n is an odd perfect number then n has the following form

$$n = p^s m^2$$

where p is prime of the form $4k + 1$, s is a positive integer of the form $4h + 1$, and $m \in \mathbb{Z}^+$, m is not divisible by p . Also, find all $n \in \mathbb{Z}^+$, $n > 1$ such that $n - 1$ and $\frac{n(n+1)}{2}$ is a perfect number.

Originally Problem 3, Day 2 of Vietnam National Olympiad.

We received only 1 solution. We present the solution by David Manes.

Let $n = p_1^{s_1} p_2^{s_2} \dots p_r^{s_r}$ be the prime factorization of the odd integer n and assume that n is perfect. Then $2n = \sigma(n) = \sigma(p_1^{s_1})\sigma(p_2^{s_2}) \dots \sigma(p_r^{s_r})$. Note that $2n \equiv 2 \pmod{4}$ since n is odd implies either $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$. Therefore, $\sigma(n) = 2n$ is divisible by 2, but not by 4. Hence, exactly one of the $\sigma(p_i^{s_i})$, say $\sigma(p_1^{s_1})$, is an even integer (but $\sigma(p_1^{s_1})$ is not divisible by 4) and the remaining $\sigma(p_i^{s_i})$ are odd integers.

Given a prime p_i , either $p_i \equiv 1 \pmod{4}$ or $p_i \equiv 3 \pmod{4}$. If $p_i \equiv 3 \equiv -1 \pmod{4}$, then

$$\sigma(p_i^{s_i}) = 1 + p_i + p_i^2 + \dots + p_i^{s_i} \equiv 1 + (-1) + (-1)^2 + \dots + (-1)^{s_i} \pmod{4}.$$

Therefore,

$$\sigma(p_i^{s_i}) \equiv \begin{cases} 0 \pmod{4}, & \text{if } s_i \text{ is odd,} \\ 1 \pmod{4}. & \text{if } s_i \text{ is even.} \end{cases}$$

Thus, $p_1 \not\equiv 3 \pmod{4}$ since $\sigma(p_1^{s_1}) \equiv 2 \pmod{4}$. Hence, $p_1 = p \equiv 1 \pmod{4}$. Furthermore, $\sigma(p_i^{s_i}) \equiv 0 \pmod{4}$ ($i > 1$) implies that 4 divides $\sigma(p_i^{s_i})$, a contradiction since $\sigma(n)$ is divisible by 2, but not by 4. Therefore, if $p_i \equiv 3 \pmod{4}$ for some $i > 1$, then its exponent s_i is an even integer. On the other hand, if the prime $p_i \equiv 1 \pmod{4}$, then

$$\begin{aligned} \sigma(p_i^{s_i}) &= 1 + p_i + p_i^2 + \dots + p_i^{s_i} \\ &\equiv 1 + 1^1 + 1^2 + \dots + 1^{s_i} \pmod{4} \\ &\equiv s_i + 1 \pmod{4}. \end{aligned}$$

The statement $\sigma(p_1^{s_1}) \equiv 2 \pmod{4}$ implies $s_1 \equiv 1 \pmod{4}$. For the remaining values of i , we have $\sigma(p_i^{s_i}) \equiv 1$ or $3 \pmod{4}$, and therefore $s_i \equiv 0$ or $2 \pmod{4}$; in any case, s_i is an even integer. Hence, regardless of whether $p_i \equiv 1$ or $3 \pmod{4}$, the exponent s_i is always even for $i \neq 1$. Therefore, an odd perfect number n can be expressed as

$$n = p^s p_2^{2h_2} \dots p_r^{2h_r} = p^s (p_2^{h_2} \dots p_r^{h_r})^2 = p^s m^2,$$

where $m = p_2^{h_2} \dots p_r^{h_r}$, $p \equiv 1 \pmod{4}$ is a prime of the form $4k + 1$, $s \equiv 1 \pmod{4}$ is an integer of the form $4h + 1$, p is not a divisor of m by the prime factorization of n and $n \equiv 1 \pmod{4}$ since m is an odd integer implies $m^2 \equiv 1 \pmod{4}$.

For the last part of the problem, we will try to show that $n = 7$ is the only positive integer such that $n - 1 = 6$ and $n(n + 1)/2 = 28$ are perfect numbers. Consider the following cases that correspond to the four parity cases of $n - 1$ and $n(n + 1)/2$.

Case 1: $n \equiv 0 \pmod{4}$. Then $n - 1 \equiv -1 \equiv 3 \pmod{4}$ implies $n - 1$ is not perfect by the solution to the first part of the problem.

Case 2: $n \equiv 1 \pmod{4}$. Then $n - 1$ is even and $n(n + 1)/2$ is odd. Assume both integers are perfect. Then, by Euler's theorem, $n - 1 = 2^{p-1}(2^p - 1)$ where p and $2^p - 1$ are odd primes. Then

$$n(n + 1)/2 = (2^{2p-1} - 2^{p-1} + 1)(2^{2p-2} - 2^{p-2} + 1).$$

Both of these factors are relatively prime and the above solution requires that one of these terms is a square. Since p is odd,

$$2^{2p-1} - 2^{p-1} + 1 \equiv 2 \pmod{3}$$

and so is not a square. The other factor lies between two consecutive squares; that is,

$$(2^{p-1} - 1)^2 < 2^{2p-2} - 2^{p-2} + 1 < (2^{p-1})^2.$$

Therefore, neither factor is a square so that $n(n+1)/2$ is not perfect.

Case 3: $n \equiv 2 \pmod{4}$. Then $n+1 \equiv 3 \pmod{4}$ and therefore has a prime factor q congruent to 3 modulo 4 with an odd exponent in the prime factorization of $n+1$. Since n and $n+1$ are relatively prime, q has the same odd exponent in the prime factorization of $n(n+1)/2$. By the above solution to the first part, $n(n+1)/2$ cannot be perfect.

Case 4: $n \equiv 3 \pmod{4}$. Then $n-1$ is divisible by 2, but not by 4. The only even perfect number not divisible by 4 is 6. Therefore $n-1$ is perfect only when $n=7$. Also, $n(n+1)/2 = 28$ is perfect.

OC359. Let a, b, c, d be positive numbers such that $a + b + c + d = 3$. Prove

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} \leq \frac{1}{a^3 b^3 c^3 d^3}.$$

Originally Problem 7, Grade 11, Day 2 of 2016 All-Russian Olympiad.

We received 3 solutions. We present the solution by Oliver Geupel.

Without loss of generality assume $0 < a \leq b \leq c \leq d$. We successively obtain

$$\begin{aligned} a^3 b^3 c^3 &\leq a^3 b^3 d^3 \leq a^2 b c^3 d^3 \leq a b^2 c^3 d^3, \\ a^3 b^3 d^3 + a^3 b^3 c^3 &\leq 3 a^2 b c^3 d^3 + 3 a b^2 c^3 d^3, \\ b^3 c^3 d^3 + a^3 c^3 d^3 + a^3 b^3 d^3 + a^3 b^3 c^3 &\leq (a^3 + 3 a^2 b + 3 a b^2 + b^3) c^3 d^3 \\ &= (a+b)^3 c^3 d^3. \end{aligned} \tag{1}$$

By the geometric mean - arithmetic mean inequality and the hypothesis, it holds

$$(a+b)^3 c^3 d^3 \leq \left(\frac{(a+b) + c + d}{3} \right)^9 = 1. \tag{2}$$

Combining (1) and (2), we successively conclude

$$\begin{aligned} b^3 c^3 d^3 + a^3 c^3 d^3 + a^3 b^3 d^3 + a^3 b^3 c^3 &\leq 1, \\ \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} &\leq \frac{1}{a^3 b^3 c^3 d^3}. \end{aligned}$$

This completes the proof.

OC360. Let A, B and F be positive integers with $A < B < 2A$. A flea is at the number 0 on the number line. The flea can move by jumping to the right by A or by B . Before the flea starts jumping, Lavaman chooses finitely many intervals $\{m+1, m+2, \dots, m+A\}$ consisting of A consecutive positive integers, and places lava at all of the integers in the intervals. The intervals must be chosen so that:

1. any two distinct intervals are disjoint and not adjacent;
2. there are at least F positive integers with no lava between any two intervals;
3. no lava is placed at any integer less than F .

Prove that the smallest F for which the flea can jump over all the intervals and avoid all the lava, regardless of what Lavaman does, is $F = (n-1)A + B$, where n is the positive integer such that $\frac{A}{n+1} \leq B - A < \frac{A}{n}$.

Originally Problem 4 of 2016 Canadian Mathematical Olympiad.

We received no solutions to this problem.

OC361. Let $n \geq 2$ be a positive integer and define k to be the number of primes less than or equal to n . Let A be a subset of $S = \{2, \dots, n\}$ such that $|A| \leq k$ and no two elements in A divide each other. Show that one can find a set B of cardinality k such that $A \subseteq B \subseteq S$ and no two elements in B divide each other.

Originally Problem 4 of Day 2 of 2016 China National Olympiad.

We received only one submission. We present the solution by Mohammed Aassila.

We prove the result by backward induction on $|A|$.

For $|A| = k$, take $B = A$.

Now, suppose that the result is true for $|A| = m < k$ and we will prove it true for $|A| = m - 1$. Let $\{p_1, p_2, \dots, p_k\}$ be the set of all primes less than or equal to n and for $i = 1, 2, \dots, k$ define

$$A_i = \{x \in A : p_i \text{ is the largest prime divisor of } x\}.$$

Note that A is a disjoint union of the A_i 's and $|A_i| < |A|$. If all the A_i 's are nonempty, then

$$m - 1 = |A| = \sum_{i=1}^k |A_i| \geq k,$$

contradiction. So, A_j is empty for some $1 \leq j \leq k$. Let $p = p_j$ and let p^α be the largest power of $p \leq n$. We claim that A contains no divisors nor multiples of p^α . Indeed, if $p^\beta \in A$ for some $\beta \leq \alpha$, it would follow that $p^\beta \in A_j$, contradiction. Meanwhile, if $cp^\alpha \in A$ for some $c > 1$, then cp^α has a prime divisor $q > p$ because $cp^\alpha \notin A_j$. Hence, $q \mid c$, which implies that $cp^\alpha \geq qp^\alpha > p^{\alpha+1} > n$, contradiction. Thus, if $A' = A \cup \{p^\alpha\}$, then no two distinct elements of A' divide one another.

Since $|A'| = m$ and $A \subseteq A'$, we can then find the desired B by the inductive hypothesis applied to A' .

OC362. Given a positive prime number p , prove that there exists a positive integer α such that $p \mid \alpha(\alpha - 1) + 3$ if and only if there exists a positive integer β such that $p \mid \beta(\beta - 1) + 25$.

Originally Problem 2 of Day 1 of 2016 Spain Mathematical Olympiad.

We received 4 submissions. We present 2 solutions.

Solution 1, by Oliver Geupel.

The assertion holds true for $p = 3$, because $\alpha = 3$ and $\beta = 2$ are integers with the required properties. In what follows we assume $p \neq 3$.

First, suppose that there exists a positive integer α such that $p \mid \alpha(\alpha - 1) + 3$. We prove that there exists a positive integer β such that $p \mid \beta(\beta - 1) + 25$. Specifically, $\beta = 3\alpha - 1$ has the required property. In fact,

$$\beta(\beta - 1) + 25 = (3\alpha - 1)(3\alpha - 2) + 25 = 9(\alpha(\alpha - 1) + 3).$$

Next, suppose that there exists a positive integer β such that $p \mid \beta(\beta - 1) + 25$. We have to show that there exists a positive integer α such that $p \mid \alpha(\alpha - 1) + 3$. Observe that if an integer α has the property that $p \mid \alpha(\alpha - 1) + 3$, then every integer that is congruent to α modulo p , has the same property. As a consequence, it is enough to determine a (not necessarily positive) integer α such that $p \mid \alpha(\alpha - 1) + 3$. Let $\omega = 3^{p-2}(2\beta - 1)$. We prove that $\alpha = \beta - \omega$ has the required property.

Applying Fermat's Little Theorem, we see that

$$\begin{aligned} \omega^2 &\equiv 3^{2p-4} (4(\beta(\beta - 1) + 25) - 99) \\ &\equiv 3^{2p-4} \cdot (-99) \\ &\equiv (3^{p-1})^2 \cdot (-11) \\ &\equiv -11 \pmod{p}. \end{aligned}$$

$$\begin{aligned} \text{Also, } \omega(2\beta - 1) &\equiv 3^{p-2} (4(\beta(\beta - 1) + 25) - 99) \\ &\equiv 3^{p-2} \cdot (-99) \\ &\equiv 3^{p-1} \cdot (-33) \\ &\equiv -33 \pmod{p}. \end{aligned}$$

$$\begin{aligned} \text{Consequently, } \alpha(\alpha - 1) + 3 &\equiv (\beta - \omega)(\beta - \omega - 1) + 3 \\ &\equiv \beta(\beta - 1) + \omega^2 + \omega(1 - 2\beta) + 3 \\ &\equiv -25 - 11 + 33 + 3 \\ &\equiv 0 \pmod{p}. \end{aligned}$$

Hence α has the required property.

Solution 2, by David Manes.

Note that $p \neq 2$ since

$$\alpha(\alpha - 1) + 3 \equiv \beta(\beta - 1) + 25 \equiv 1 \pmod{2}$$

for all positive integers α and β . Moreover, if $p = 11$, then $\alpha = \beta = 6$ and if $p = 3$, then $\alpha = 1$ and $\beta = 2$. Let $p \neq 3, 11$ be an odd prime and assume there exists a positive integer α such that $p \mid \alpha(\alpha - 1) + 3$. Then $\alpha^2 - \alpha + 3 \equiv 0 \pmod{p}$. Multiplying by 4 and simplifying, we obtain $(2\alpha - 1)^2 \equiv -11 \pmod{p}$. Therefore, -11 is a quadratic residue of p so that the Legendre symbol

$$\left(\frac{-11}{p}\right) = \left(\frac{-1}{p}\right) \left(\frac{11}{p}\right) = 1.$$

Consequently, -99 is also a quadratic residue of p since the Legendre symbol

$$\left(\frac{-99}{p}\right) = \left(\frac{-1}{p}\right) \left(\frac{11}{p}\right) \left(\frac{9}{p}\right) = 1.$$

Note that $(9/q) = 1$ for all odd primes $q \neq 3$. Thus, there exists a positive integer γ such that $\gamma^2 \equiv -99 \pmod{p}$. Solving the linear congruence $2x - 1 \equiv \gamma \pmod{p}$ for x a least residue of p produces the positive integer β . Therefore, $2\beta - 1 \equiv \gamma \pmod{p} \implies (2\beta - 1)^2 \equiv \gamma^2 \equiv -99 \pmod{p} \implies 4\beta^2 - 4\beta + 100 \equiv 0 \pmod{p}$. Since $p \neq 2$, it follows that 4 and p are relatively prime so that dividing the last congruence by 4, we obtain $\beta^2 - \beta + 25 \equiv 0 \pmod{p}$. Hence, there exists a positive integer β such that $p \mid \beta(\beta - 1) + 25$. The converse now follows by starting with the assumption for β and following the same steps that we used for α and using the fact that if -99 is a quadratic residue of the odd prime p , then so is -11 .

Editor's Comments. The Missouri State University Problem Solving Group solved the generalization of the problem, i.e. the result holds for all p , whether prime or not. We give the idea of the proof, so that the reader can fill all the details. One implication is easy, because if there exists α such that $\alpha^2 - \alpha + 3 \equiv 0 \pmod{p}$, then taking $\beta = 3\alpha - 1$, we have

$$\beta^2 - \beta + 25 = 9(\alpha^2 - \alpha + 3) \equiv 0 \pmod{p}.$$

Conversely, if $p \not\equiv 0 \pmod{3}$, then there exists γ such that $3\gamma \equiv 1 \pmod{p}$. Taking $\alpha = \gamma(\beta + 1)$, the reader can verify that $\alpha^2 - \alpha + 3 \equiv 0 \pmod{p}$. Now, one can prove by induction on $m > 0$ that the congruence $t^2 - t + 3 \equiv 0 \pmod{3^m}$ has a solution with $t \equiv 0 \pmod{3}$ for all $m > 0$ (if $t = 3k$ is such that $t^2 - t + 3 = 3^m \ell$, letting $s = t + 3^m \ell$, it's easy to see that $s^2 - s + 3 \equiv 0 \pmod{3^{m+1}}$). Now, if $p \equiv 0 \pmod{3}$, then $p = 3^m n$ with $m > 0$ and $n \not\equiv 0 \pmod{3}$. If $\beta^2 - \beta + 25 \equiv 0 \pmod{p}$, then $\beta^2 - \beta + 25 \equiv 0 \pmod{n}$. Since $n \not\equiv 0 \pmod{3}$, then there is a such that $a^2 - a + 3 \equiv 0 \pmod{n}$ and there is also t such that $t^2 - t + 3 \equiv 0 \pmod{3^m}$. Now it is sufficient to use Chinese Remainder Theorem to conclude that there is α such that $\alpha \equiv a \pmod{n}$ and $\alpha \equiv t \pmod{3^m}$, i.e. $\alpha^2 - \alpha + 3 \equiv 0 \pmod{n}$ and $\alpha^2 - \alpha + 3 \equiv 0 \pmod{3^m}$ and the conclusion follows.

OC363. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$f(y)f(x + f(y)) = f(x)f(xy)$$

for all positive real numbers x and y .

Originally Algebra Problem 3 of 3rd Round of 2016 Iranian Mathematical Olympiad.

We received only one submission. We present the solution by Mohammed Aassila.

Let $f(1) = a$ and let $P(x, y)$ be the assertion

$$f(y)f(x + f(y)) = f(x)f(xy).$$

Then,

$$P(x, 1) \implies f(x + a) = \frac{(f(x))^2}{a}$$

and

$$\begin{aligned} P(x + a, y) &\implies \frac{f(y)(f(x + f(y)))^2}{a} = \frac{(f(x))^2 f(xy + ay)}{a} \\ &\implies f(x + f(y)) = \frac{f(x)f(xy + ay)}{f(xy)} \\ &\implies f(y)f(xy + ay) = (f(xy))^2. \end{aligned}$$

Hence,

$$((f(2x))^2 = f(2)f(2x + 2a) = \frac{f(2)(f(2x + a))^2}{a} = \frac{f(2)(f(2x))^4}{a^3},$$

which gives

$$((f(2x))^2 = \frac{a^3}{f(2)},$$

which means that $f(x)$ is a constant function.

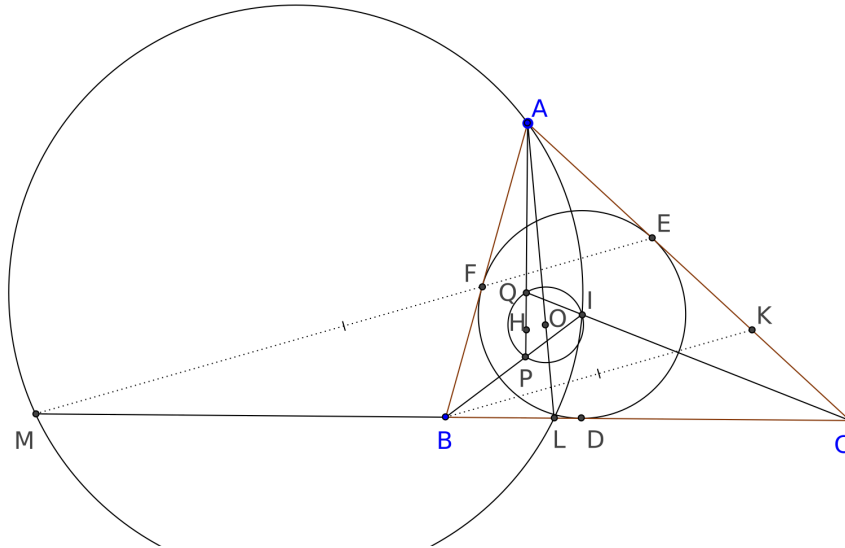
Hence, all solutions are $f(x) \equiv c$ for some constant $c > 0$ which clearly satisfies the functional equation.

OC364. Consider an acute triangle ABC . Suppose $AB < AC$, let I be the incenter, D the foot of perpendicular from I to BC , and suppose that altitude AH meets BI and CI at P and Q , respectively. Let O be the circumcenter of $\triangle IPQ$, extend AO to meet BC at L and suppose that the circumcircle of $\triangle AIL$ meets BC again at N . Prove that

$$\frac{BD}{CD} = \frac{BN}{CN}.$$

Originally Problem 7 of Day 2 of 2016 China Girls Mathematical Olympiad.

We received only one submission. We present the solution by Oliver Geupel.



Let E and F denote the feet of perpendiculars from I to CA and AB , respectively. Straightforward angle chasing gives us

$$\angle EDF = \angle PIQ = 90^\circ - \angle A/2 \quad \text{and} \quad \angle FED = \angle QPI = 90^\circ - \angle B/2.$$

Hence the triangles DEF and IPQ are similar.

Further, we can find that $\angle AIQ = \angle IPQ = 90^\circ - \angle B/2$. Thus AI is tangent to the circumcircle of triangle IPQ . As a consequence, triangle PAI is similar to triangle EMD , where M is the point of intersection of BC and EF . By similarity, we have $\angle LMI = \angle DMI = \angle IAO = \angle IAL$. Hence, M is the point of intersection of BC and the circumcircle of triangle AIL , that is, $M = N$.

Suppose that the parallel to EM through B meets AC at point K . We have $\angle BKC = \angle FEC = \angle BFE = 180^\circ - \angle MFB$. Let a , b , c , and s denote the lengths of sides and the semiperimeter, respectively, of triangle ABC . We obtain

$$\frac{BN}{s-b} = \frac{BM}{BF} = \frac{\sin \angle MFB}{\sin \angle BMF} = \frac{\sin \angle BKC}{\sin \angle CBK} = \frac{BC}{CK} = \frac{a}{b-c}.$$

Consequently,

$$\begin{aligned} BN &= \frac{a(s-b)}{b-c}, \\ CN &= BC + BN = a + \frac{a(s-b)}{b-c} = \frac{a(s-c)}{b-c}, \\ \frac{BD}{CD} &= \frac{s-b}{s-c} = \frac{BN}{CN}. \end{aligned}$$

The proof is complete.

OC365. A square $ABCD$ is divided into n^2 equal small (fundamental) squares by drawing lines parallel to its sides. The vertices of the fundamental squares are called vertices of the grid. A rhombus is called *nice* when:

1. it is not a square;
2. its vertices are points of the grid;
3. its diagonals are parallel to the sides of the square $ABCD$.

Find (as a function of n) the number of nice rhombuses (n is a positive integer greater than 2).

Originally Problem 4 of 2016 Greece National Olympiad Problem.

We received 2 correct submissions. We present the solution by the Missouri State University Problem Solving Group.

Let $A = (0, 0)$, $B = (n, 0)$, $C = (n, n)$, and $D = (0, n)$, so the vertices of the small squares have integer coordinates. We claim that the number of nice rhombuses centered at the point (i, j) is

$$xy - \min(x, y), \text{ where } x = \min(i, n - i) \text{ and } y = \min(j, n - j)$$

(i.e. x and y are the distances from the point to sides of $ABCD$ nearest it). To see this, note that a rhombus satisfying conditions (2) and (3) and centered at (i, j) has vertices

$$(i + s, j), (i, j + t), (i - s, j) \text{ and } (i, j - t)$$

with $1 \leq s \leq x$ and $1 \leq t \leq y$. The fact that $x = \min(i, n - i)$ and $y = \min(j, n - j)$ guarantees that the vertices are points of the grid. Therefore the number of rhombuses satisfying conditions (2) and (3) and centered at (i, j) is xy . The rhombuses of this type that are squares have $s = t$ and there are $\min(x, y)$ of these. The claim follows.

We first deal with the case $n = 2k + 1$. Here we can sum over all $1 \leq i, j \leq k$ and multiply by 4 to obtain the final answer. Therefore, the number of nice rhombuses is

$$\begin{aligned} 4 \left(\sum_{i=1}^k \sum_{j=1}^k (ij - \min(i, j)) \right) &= 4 \left(\sum_{i=1}^k \sum_{j=1}^k ij - \sum_{i=1}^k \sum_{j=1}^k \min(i, j) \right) \\ &= 4 \left(\sum_{i=1}^k i \sum_{j=1}^k j - \sum_{t=1}^k \sum_{i=t}^k \sum_{j=t}^k 1 \right) \\ &= 4 \left(\left(\frac{k(k+1)}{2} \right)^2 - \sum_{t=1}^k (k+1-t)^2 \right) \\ &= 4 \left(\frac{k^2(k+1)^2}{4} - \frac{k(k+1)(2k+1)}{6} \right) \\ &= \frac{(3k+2)(k+1)k(k-1)}{3}. \end{aligned}$$

When $n = 2k$, summing over all $1 \leq i, j \leq k - 1$ and over $i = k, 1 \leq j \leq k - 1$ multiplying this result by 4 and adding the value for $i = j = k$ gives the total number of nice rhombuses. In this case, we obtain

$$\begin{aligned} & 4 \left(\sum_{i=1}^{k-1} \sum_{j=1}^{k-1} (ij - \min(i, j)) + \sum_{j=1}^{k-1} (kj - j) \right) + k^2 - k \\ &= \frac{(3k-1)k(k-1)(k-2)}{3} + 4(k-1) \frac{k(k-1)}{2} + k^2 - k \\ &= \frac{k(k-1)(3k^2 - k - 1)}{3} \end{aligned}$$

as the number of nice rhombuses.

