MathemAttic

No. 2

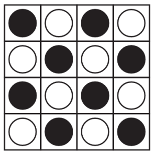
The problems in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **May 15, 2019**.

MA6. A rectangular sheet of paper is labelled $ABCD$, with AB being one of the longer sides. The sheet is folded so that vertex A is placed exactly on top of the opposite vertex C. The fold line is XY, where X lies on AB and Y lies on CD. Prove that the triangle CXY is isosceles.

MA7. Sixteen counters, which are black on one side and white on the other, are arranged in a 4 by 4 square. Initially all the counters are facing black side up. In one 'move', you must choose a 2 by 2 square within the square and turn all four counters over once. Describe a sequence of 'moves' of minimum length that finishes with the visible colours of the counters of the 4 by 4 square alternating (as shown in the diagram).



MA8. I have two types of square tile. One type has a side length of 1 cm and the other has a side length of 2 cm. What is the smallest square that can be made with equal numbers of each type of tile?

MA9. The letters a, b, c, d, e and f represent single digits and each letter represents a different digit. They satisfy the following equations:

$$a + b = d, \quad b + c = e, \quad d + e = f.$$

Find all possible solutions for the values of a, b, c, d, e and f.

MA10. An arithmetic and a geometric sequence, both consisting of only positive integral terms, share the same first two terms. Show that each term of the geometric sequence is also a term of the arithmetic sequence.

Les problémes dans cette section sont appropriés aux étudiants de l'école secondaire.

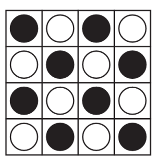
Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mai 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA6. Une feuille de papier rectangulaire $ABCD$ est telle que AB est un de ses longs côtés. Cette feuille est pliée de façon à ce que le coin A se trouve directement par dessus le coin opposé C. La ligne de pli est dénotée XY, où X se trouve sur le côté AB et Y se trouve sur le côté CD. Démontrer que le triangle CXY est isocèle.

MA7. Seize jetons, noirs d'un côté et blancs de l'autre, sont disposés sur une grille 4 par 4. Au départ, chaque jeton montre sa face noire. À chaque "tour", on choisit un carré 2 par 2 dans le carré original et on y renverse tous ses jetons. Déterminer une suite de "tours" de longueur minimale telle que les couleurs visibles des jetons deviennent alternantes (voir schéma).



MA8. On dispose de deux tailles de tuiles carrées. La première sorte est de côtés 1 cm, tandis que la deuxième sorte a des côtés de 2 cm. Quel est le plus petit carré qui puisse être formé à l'aide de nombres égaux des deux sortes de tuile ?

MA9. Les lettres a, b, c, d, e et f dénotent des entiers à un chiffre, chaque lettre correspondant à un entier différent. De plus, ces lettres satisfont aux équations suivantes :

$$a + b = d, \quad b + c = e, \quad d + e = f.$$

Déterminer toutes les solutions possibles a, b, c, d, e et f.

MA10. Une suite arithmétique et une suite géométrique, consistant d'entiers positifs, partagent les mêmes deux premiers termes. Démontrer que chaque terme de la progression géométrique est aussi un des termes de la progression arithmétique.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(3), p. 92–93; and 44(4), p. 133–136.

CC311. Suppose $1 \leq a < b < c < d \leq 100$ are four natural numbers. What is the minimum possible value of $\frac{a}{b} + \frac{c}{d}$?

Originally Problem 1 from Group round of 2016 Georgia Tech High School Mathematics Competition.

We received 14 submissions. We present two submissions.

Solution 1, by Ángel Plaza.

Since $1 \leq a < b < c < d \leq 100$, the minimum value is attained for $a = 1$ and $d = 100$ (otherwise the solution can be made smaller by making a smaller or d larger). Also since $b < c$, we may assume that $c = b + 1$ because otherwise if $c = b + 2$ or greater than that the result will be greater than for $c = b + 1$.

The problem is now an optimization problem with

$$f = \frac{1}{x} + \frac{x+1}{100}, \quad f'(x) = \frac{-1}{x^2} + \frac{1}{100} \quad \text{and} \quad f''(x) = \frac{2}{x^3} > 0$$

for $x > 0$. This implies that function $f(x)$ present a minimum value at $x = 10$ which is the root of $f'(x) = 0$, with $f(10) = 0.21$ and the problem is done.

Solution 2, by Digby Smith.

As above, we see that $a = 1$, $d = 100$, and $c = b + 1$. We will minimize

$$\frac{1}{b} + \frac{b+1}{100} = \left(\frac{1}{b} + \frac{b}{100}\right) + \frac{1}{100}.$$

Applying the AM-GM inequality to the bracketed terms, we get

$$\frac{1}{b} + \frac{b}{100} \geq 2\sqrt{\frac{1}{b} \cdot \frac{b}{100}} \geq \frac{2}{10}.$$

Then

$$\left(\frac{1}{b} + \frac{b}{100}\right) + \frac{1}{100} \geq \left(\frac{2}{10}\right) + \frac{1}{100} \geq \frac{21}{100}.$$

Equality holds for $b = 10$.

It then follows that the minimum value is $\frac{a}{b} + \frac{c}{d} = \frac{21}{100}$ when $a = 1$, $b = 10$, $c = 11$, and $d = 100$.

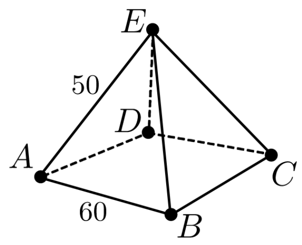
CC312. Choose four points A, B, C and D on a circle uniformly at random. What is the probability that the lines AB and CD intersect outside the circle?

Originally Problem 8 from Fillins round of 2016 Georgia Tech High School Mathematics Competition.

We received 7 correct solutions and one incorrect solution. We present the solution by Kathleen Lewis.

Since the points are chosen uniformly at random, any set of 4 points is as likely to be labelled in any one of the 24 ways as in any other way. But once A is labeled, there are four ways to label B, C and D for which the lines AB and CD intersect outside the circle and only two for which they intersect inside the circle. So the probability of intersecting outside is $2/3$.

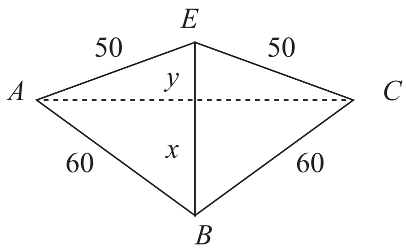
CC313. Consider a pyramid whose faces consist of a 60×60 square base $ABCD$ and four $60-50-50$ triangles that join at the apex E. If you are only allowed to move on the surfaces of the four triangles, what is the length of the shortest path between A and C?



Originally Problem 10 from Fillins round of 2016 Georgia Tech High School Mathematics Competition.

We received 11 submissions. We present the solution by Carlos Moreno and Ángel Plaza.

Imagine flattening two triangular faces of the pyramid as shown.



Since $EB = 50$, let $x + y = 50$ as shown. By the Pythagorean theorem, it is deduced $x = 36$, $y = 14$. Then the height over EB in triangle ABC is $h = 48$ and consequently the shortest path between A and C is $2 \times 48 = 96$.

CC314. An infinite sequence $a_1, a_2, \dots$ of 1's and 2's is uniquely defined by the following properties:

1. $a_1 = 1$ and $a_2 = 2$,
2. For every $n \geq 1$, the number of 1's between the nth 2 and the $(n+1)^{\text{st}}$ 2 is equal to a_{n+1}.

Is the sequence periodic from the beginning?

Originally Problem 4 from Proof round of 2016 Georgia Tech High School Mathematics Competition.

We received 4 submissions of which 2 were both correct and complete. Both correct solutions used proof-by-contradiction strategy. We present the solution by Steven Chow.

Assume for the sake of contradiction that the sequence is periodic from the beginning. Let k be the fundamental period of the sequence. Let t be the number of 2's in $a_1, a_2, \dots, a_k$. Due to periodicity, t is the number of 2's in any subsequence of length k.

We claim that for all integers $j \geq 2$, $a_j = a_{j+t}$. Indeed a_j is the number of 1's between the $(j-1)^{\text{st}}$ 2 and the j^{th} 2, and a_{j+t} is the number of 1's between the $(j+t-1)^{\text{st}}$ 2 and the $(j+t)^{\text{th}}$ 2. Between the $(j-1)^{\text{st}}$ 2 and the $(j+t-1)^{\text{st}}$ 2, including the $(j-1)^{\text{st}}$ 2 and excluding the $(j+t-1)^{\text{st}}$ 2, there are exactly t 2's. The sequence is periodic with exactly t 2's in any subsequence of length equal to the fundamental period. Therefore the subsequence starting at the $(j+t-1)^{\text{st}}$ 2 must be a repeat of the subsequence starting at the $(j-1)^{\text{st}}$ 2. Therefore $a_j = a_{j+t}$.

Since $a_j = a_{j+t}$ for all integers $j \geq 2$, the sequence $a_2, a_3, \dots$ is periodic with period t. However, the fundamental period is k, so $k \leq t$. From the definition of t, $t \leq k$. Therefore $t = k$ and the sequence consists of 2's only.

This is a contradiction. Therefore, the sequence is not periodic from the beginning.

Editor's Comment. The above solution can be slightly modified to obtain that the sequence is not periodic starting at n, for any $n \geq 1$.

CC315. A square table is divided into a 3×3 grid with every cell having 3 coins. In every step of a game, Terry can take 2 coins from the table as long as they come from distinct but adjacent cells. (Here "adjacent" means the two cells share a common edge.) At most how many coins can Terry take?

Originally Problem 9 from Fillins round of 2016 Georgia Tech High School Mathematics Competition.

We received 7 submissions. We present the solution by the Missouri State University Problem Solving Group.

The answer is 24 coins.

If we apply the usual checkerboard colouring with the central and corner cells coloured black and the side squares coloured white, there are 15 coins in black cells and 12 coins in white ones. But every time a pair of coins is chosen, one must be from a black cell and one from a white one. At best this will leave us with 3 coins remaining (i.e. 24 coins taken).

This result can actually be obtained by choosing pairs of coins from adjacent cells on the left-hand side of the first three rows and from two adjacent cells in the last column.

A similar argument shows that if k coins are placed in each cell of an $m \times n$ grid with m and n both odd, then at most $k(mn-1)$ coins can be taken.

Editor's Comments. There was quite a variety of solutions here. Many did *one* of the following:

- Using a checkerboard argument, showed that it is impossible to take more than 24 coins.
- By construction, showed that it is possible to take exactly 24 coins.

Only a few showed both sides of the proof.

CC316. Positive integers (x, y, z) form a *Trenti-triple* if $3x = 5y = 2z$. Show that for every Trenti-triple (x, y, z) the product xyz must be divisible by 900.

Originally Question 3 from Hypatia 2011.

We received 13 submissions, all correct. We feature a composite solution based on essentially the same ones given by most of these solvers.

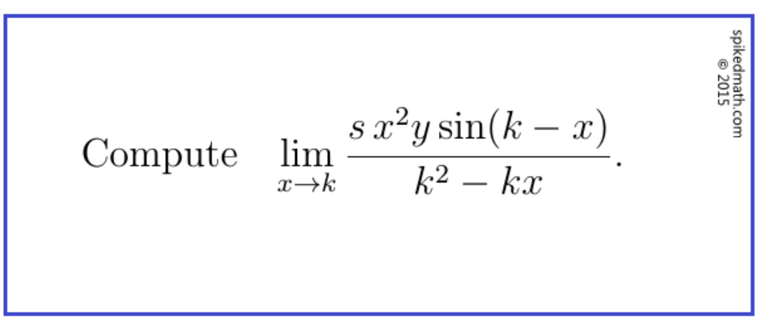
Note first that 3, 5, and 2 are mutually relatively co-prime.

Hence, from $3x = 5y = 2z$ we deduce that $5 \mid x$ and $2 \mid x$, so $10 \mid x$.

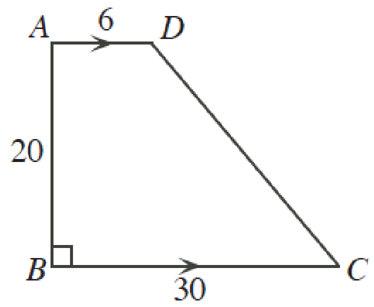
Similarly, from $5y = 2z = 3x$ we deduce that $2 \mid y$ and $3 \mid y$, so $6 \mid y$.

Finally, from $2z = 3x = 5y$ we deduce that $3 \mid z$ and $5 \mid z$, so $15 \mid z$.

It then follows that xyz is divisible by $10 \cdot 6 \cdot 15$ or 900.



CC317. In the diagram, $ABCD$ is a trapezoid with AD parallel to BC and BC perpendicular to AB. Also, $AD = 6$, $AB = 20$, and $BC = 30$.



1. Determine the area of trapezoid $ABCD$.
2. There is a point K on AB such that the area of triangle KBC equals the area of quadrilateral $KADC$. Determine the length of BK.
3. There is a point M on DC such that the area of triangle MBC equals the area of quadrilateral $MBAD$. Determine the length of MC.

Originally Question 3 from Hypatia 2008.

We received 8 correct solutions. We present the solution by Ricard Peiró i Estruch.

1. The area of $ABCD$ is $\frac{30+6}{2} \cdot 20 = 360$.

2. Set $x = BK$. The area of triangle KBC is half the area of trapezoid $ABCD$, so

$$\frac{1}{2} \cdot 30x = \frac{1}{2} \cdot 360.$$

Solving gives $x = 12$.

3. Let D' be the projection of D onto line BC. Then $DD' = AB = 20$ and

$$D'C = BC - AD = 24.$$

Applying the Pythagorean Theorem to right triangle $DD'C$ gives

$$CD = \sqrt{24^2 + 20^2} = 4 \cdot \sqrt{61}.$$

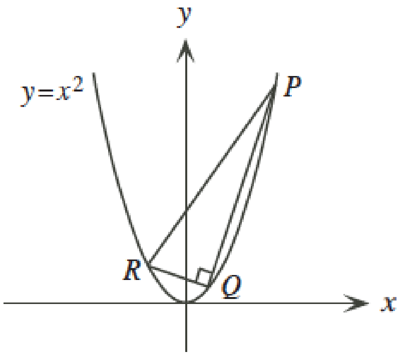
Since the area of triangle MBC is half the area of trapezoid $ABCD$ and thus equal to the area of triangle KBC, and the two triangles share base $\overline{BC}$, they have the same height. Thus $\overline{KM}$ is parallel to both $\overline{BC}$ and $\overline{AD}$. It follows that

$$\frac{MC}{BK} = \frac{CD}{AB} \quad \text{or} \quad \frac{MC}{12} = \frac{4\sqrt{61}}{20}.$$

Solving gives

$$MC = \frac{12\sqrt{61}}{5}.$$

CC318. Right-angled triangle PQR is inscribed in the parabola with equation $y = x^2$, as shown. Points P, Q and R have coordinates $(p, p^2), (q, q^2)$ and (r, r^2), respectively. If p, q and r are integers, show that $2q + p + r = 0$.



Originally Question 3c from Hypatia 2012.

We received 9 submissions of which all but one were correct and complete. We present the solution by Digby Smith.

Note from the diagram that $p > q > r$. Since $\angle RQP = 90^\circ$ it follows that QR and QP are perpendicular. Moreover, the slope of line QP is

$$m_{QP} = (p^2 - q^2)/(p - q) = p + q,$$

the slope of line QR is

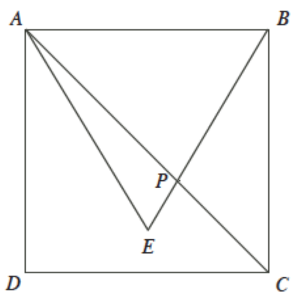
$$m_{QR} = (q^2 - r^2)/(q - r) = q + r,$$

and

$$m_{QP} m_{QR} = (p + q)(q + r) = -1.$$

Since p, q, and r are integers it then follows that either $p+q = 1$ and $q+r = -1$, or $p + q = -1$ and $q + r = 1$. In both cases, after adding the two equations, it follows that $2q + p + r = 0$.

CC319. In the diagram, square $ABCD$ has sides of length 4, and triangle ABE is equilateral. Line segments BE and AC intersect at P. Determine the exact area of triangle APE.



Originally Question 3 from Hypatia 2010.

We received 14 correct solutions from 11 solvers. The solver of one incorrect submission apparently misread the question thinking that the side length of the given square is 1. We present the solution by Richard Peiró i Estruch.

From P, draw line PH perpendicular to AB and let $\overline{AH} = x$.

Since $\angle BAP = 45°$, we have $\overline{PH} = \overline{AH} = x$.

Since

$$\overline{PH} = \overline{(BH)} \tan 60° = (4 - x)\sqrt{3},$$

we have

$$x = \sqrt{3}(4 - x).$$

Solving yields

$$x = \frac{4\sqrt{3}}{\sqrt{3}+1} = 2\sqrt{3}(\sqrt{3} - 1) = 6 - 2\sqrt{3}.$$

Finally,

$$\begin{aligned} |\triangle APE| &= |\triangle ABE| - |\triangle ABP| \\ &= \frac{1}{2}(4)(2\sqrt{3}) - \frac{1}{2}(4)(6 - 2\sqrt{3}) = 4\sqrt{3} - 4(3 - \sqrt{3}) \\ &= 8\sqrt{3} - 12. \end{aligned}$$

CC320. A sequence of m P's and n Q's with $m > n$ is called *non-predictive* if there is some point in the sequence where the number of Q's counted from the left is greater than or equal to the number of P's counted from the left. For example, if $m = 5$ and $n = 2$, the sequences $PPQQPPP$ and $QPPPQPP$ are non-predictive.

1. If $m = 7$ and $n = 2$, determine the number of non-predictive sequences that begin with P.
2. Suppose that $n = 2$. Show that for every $m > 2$, the number of non-predictive sequences that begin with P is equal to the number of non-predictive sequences that begin with Q.
3. Determine the number of non-predictive sequences with $m = 10$ and $n = 3$.

Originally Question 3 from Hypatia 2013.

We received five submissions. We present the solution by Kathleen E. Lewis.

1. The sequence must begin with PP or PQ. If it begins with PP and has only 2 Q's, then the only way that it can be non-predictive is to begin $PPQQ$. But then the rest of the terms must be P's, so the only such sequence is $PPQQPPPPP$. Now suppose that it begins with PQ. Then it's already non-predictive, so the

remaining terms can be in any order. Since there are 6 P's and one Q remaining, there are 7 possible arrangements. Therefore, there are a total of eight non-predictive sequences beginning with P.

2. Any sequence that begins with Q is automatically non-predictive. There are as many such sequences as there are arrangements of the other letters, one Q and m P's. Since there are $m+1$ places to put the Q, there are $m+1$ such sequences. The number of non-predictive sequences beginning with P can be analyzed as in part (1) above. There is only one such sequence beginning with PP, since it must start with $PPQQ$. Any sequence beginning with PQ is non-predictive, so we can arrange the other terms in any way. There are $m-1$ P's and only one Q remaining, so there are m ways to arrange them. That gives a total of $m+1$ non-predictive sequences beginning with P to match the $m+1$ beginning with Q.

3. Consider the following cases:

a) sequences starting with Q: Any sequence starting with Q is automatically non-predictive. The remaining 10 P's and 2 Q's can be arranged in any order, so there are $\binom{12}{2} = 66$ such sequences.

b) sequences starting with PQ: Again, all such sequences are non-predictive. There are 9 P's and 2 Q's still to arrange, so that gives $\binom{11}{2} = 55$ such sequences.

c) sequences starting with PPP: The only way such a sequence can be non-predictive is to have the three Q's immediately follow the first three P's, so the only non-predictive sequence of this form is $PPPQQQPPPPPPP$.

d) sequences starting with $PPQQ$: Any such sequence is non-predictive. Since there is one remaining Q to place among 8 P's, there are nine such sequences.

e) sequences starting with $PPQP$: The only way such a sequence can be non-predictive is to continue as $PPQPQQPPPPPPP$. So there is only one sequence in this case.

Total: Hence, there is a total of $66+55+1+9+1 = 132$ non-predictive sequences with 10 P's and 3 Q's.

PROBLEM SOLVING VIGNETTES

No.2

Shawn Godin

Carefully Counting Canoeists

This month, we will look at problem **7a** from the 2018 *Euclid Contest*, hosted by the The Centre for Education in Mathematics and Computing at the University of Waterloo. You can check out the contest, and past contests on the CEMC website at `www.cemc.uwaterloo.ca`.

> Eight people, including triplets Barry, Carrie and Mary, are going for a trip in four canoes. Each canoe seats two people. The eight people are to be randomly assigned to the four canoes in pairs. What is the probability that no two of Barry, Carrie and Mary will be in the same canoe?

This problem can be tricky as it involves careful counting. Since it is a probability problem we may also interpret the process of putting people into canoes in different ways, as long as we are consistent throughout our solution. We need to compute the number of ways that the people can be seated, without restriction, and the number of ways that they can be seated with the triplets separated. We will look at the problem in several different ways.

Solution #1: Completely ordered.

We will impose an "order" on the canoes and on the seats in the canoes. That is, we will consider Alice and Barry in the red canoe as different from Alice and Barry in the green canoe. Similarly, we will consider Alice and Barry in the silver canoe with Alice in front different from Alice and Barry in the silver canoe with Barry in the front. Thus there are 8 positions into which we want to order our 8 people. We can do the ordering in 8! ways.

To ensure the triplets are in separate canoes, we will assign them separately. Barry can be placed in any of the 8 seats. When it comes to seating Carrie, she cannot be in the same seat as Barry or the other empty seat in his canoe. As such, there are 6 possible seats in which to seat Carrie. Similarly, there are 4 possible seat choices for Mary. Once the triplets have been seated, we can seat the remaining 5 people in any order, which can be done in 5! ways. Thus the total number of ways to seat the people, keeping the triplets separated is $8 \times 6 \times 4 \times 5!$.

Hence, the desired probability is

$$\frac{8 \times 6 \times 4 \times 5!}{8!} = \frac{8 \times 6! \times 4}{8 \times 6! \times 7} = \frac{4}{7}.$$

Solution #2: Partially ordered.

We will impose an order on the canoes, but not on the positions within the canoes. Thus we need to pick 2 people for the first canoe, which we can do in $\binom{8}{2}$ ways. Filling the other canoes in similar ways, we get the total number of ways to fill the canoes to be $\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$. We could have similarly assumed that the 8 people were given cards randomly that had either R, G, B, or S on it, indicating that the canoeist was assigned to the red, green, blue or silver canoe, respectively. We can then think of the process as the number of ways or ordering two each of R, G, B, and S. This can be done in $\frac{8!}{(2!)^4}$ ways. It is easy to show that

$$\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{8!}{(2!)^4}$$

Next, since we want the triplets separated, we will deal with them first. First choose which canoes they will be placed into, which can be done in $\binom{4}{3}$ ways. Since we have "ordered" our canoes, the order the triplets are placed in the canoes is important. They can be placed in the chosen canoes in 3! ways. At this point we have one empty canoe, and three half filled canoes in which to place our remaining 5 canoeists. We will fill the empty canoe first, which can be done in $\binom{5}{2}$ ways. Then the remaining 3 canoeists have to be "ordered" into the three remaining canoes with the triplets, which can be done in 3! ways. Note we could have put the three canoeists with the triplets first and then filled the empty canoe with the remaining two. You may wish to show that this yields the same result. So the total number of ways to seat the people, keeping the triplets separated is $\binom{4}{3} \times (3!)^2 \times \binom{5}{2}$.

Once again, our desired probability is

$$\frac{\binom{4}{3} \times (3!)^2 \times \binom{5}{2}}{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}} = \frac{4}{7}.$$

Solution #3: Completely unordered.

In this solution we will not impose an order on either the canoes or the positions. From solution #2, the number of ways to place the canoeists in the boats where the boats are "ordered" but the seats are not is $\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$. If we want the boats not to be ordered, we need to "remove" the order of the four boats by noticing that each possible set of groupings of pairs can be assigned to the "ordered" canoes in 4! ways. Thus the number of unordered assignments is $\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{4!}$.

To assign the pairs with the triplets separated, we can assume that each triplet, in turn, will select a person with whom they will be paired. This can be done in $5 \times 4 \times 3$ ways. The remaining two people form the final pair. Thus the desired probability is

$$\frac{5 \times 4 \times 3}{\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{4!}} = \frac{4}{7}$$

Solving a problem in different ways gives you different insights. Solution #3 has the most straightforward way of calculating the number of ways to seat the canoeists with the triplets separated. It also seems like the most "real world" way of thinking of the problem (i.e. canoeists probably were assigned a partner, not a particular canoe or seating order). On the other hand, determining the number of ways to seat the canoeists without any order, becomes a little trickier. Solution #2 seems like a natural way to think about the problem, and was the way I first thought of it. Even if we are not assigning groups to particular boats, if we pick the groups by picking names out of a hat, the *process* imposes an order on the groups, even if that order is removed later on. Solution #1, while seemingly unnatural, does provide the "cleanest" solution. When dealing with probability questions it is worth thinking about considering the problems as ordered, or not. In some cases, going against the "natural" way of thinking about the problem might provide you with a simplified solution. It is worth your time to check out the official solutions to the problem from the CEMC website. Their first solution is similar to our solution #2 with a few interesting variations. Their second solution is a clever way to determine the probability without counting the groups.

Careful counting is important when dealing with probability or counting problems. I suspect that many students who did the problem, but got it wrong, probably computed the two totals for the problem with different interpretations. I wouldn't be surprised if there were many solutions that gave

$$\frac{5 \times 4 \times 3}{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}},$$

where students were thinking that they were dealing with an unordered problem but, inadvertently, imposed an ordering to the canoes. We will revisit counting problems in future columns.