CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(3), p. 92–93; and 44(4), p. 133–136.

CC311. Suppose $1 \leq a < b < c < d \leq 100$ are four natural numbers. What is the minimum possible value of $\dfrac{a}{b} + \dfrac{c}{d}$?

Originally Problem 1 from Group round of 2016 Georgia Tech High School Mathematics Competition.

We received 14 submissions. We present two submissions.

Solution 1, by Ángel Plaza.

Since $1 \leq a < b < c < d \leq 100$, the minimum value is attained for $a = 1$ and $d = 100$ (otherwise the solution can be made smaller by making a smaller or d larger). Also since $b < c$, we may assume that $c = b + 1$ because otherwise if $c = b + 2$ or greater than that the result will be greater than for $c = b + 1$.

The problem is now an optimization problem with

$$f = \frac{1}{x} + \frac{x+1}{100}, \quad f'(x) = \frac{-1}{x^2} + \frac{1}{100} \quad \text{and} \quad f''(x) = \frac{2}{x^3} > 0$$

for $x > 0$. This implies that function $f(x)$ present a minimum value at $x = 10$ which is the root of $f'(x) = 0$, with $f(10) = 0.21$ and the problem is done.

Solution 2, by Digby Smith.

As above, we see that $a = 1$, $d = 100$, and $c = b + 1$. We will minimize

$$\frac{1}{b} + \frac{b+1}{100} = \left(\frac{1}{b} + \frac{b}{100}\right) + \frac{1}{100}.$$

Applying the AM-GM inequality to the bracketed terms, we get

$$\frac{1}{b} + \frac{b}{100} \geq 2\sqrt{\frac{1}{b} \cdot \frac{b}{100}} \geq \frac{2}{10}.$$

Then

$$\left(\frac{1}{b} + \frac{b}{100}\right) + \frac{1}{100} \geq \left(\frac{2}{10}\right) + \frac{1}{100} \geq \frac{21}{100}.$$

Equality holds for $b = 10$.

It then follows that the minimum value is $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{21}{100}$ when $a = 1$, $b = 10$, $c = 11$, and $d = 100$.

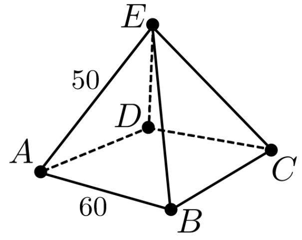
CC312. Choose four points A, B, C and D on a circle uniformly at random. What is the probability that the lines AB and CD intersect outside the circle?

Originally Problem 8 from Fillins round of 2016 Georgia Tech High School Mathematics Competition.

We received 7 correct solutions and one incorrect solution. We present the solution by Kathleen Lewis.

Since the points are chosen uniformly at random, any set of 4 points is as likely to be labelled in any one of the 24 ways as in any other way. But once A is labeled, there are four ways to label B, C and D for which the lines AB and CD intersect outside the circle and only two for which they intersect inside the circle. So the probability of intersecting outside is $2/3$.

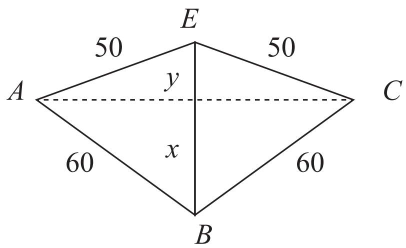
CC313. Consider a pyramid whose faces consist of a 60×60 square base $ABCD$ and four $60 - 50 - 50$ triangles that join at the apex E. If you are only allowed to move on the surfaces of the four triangles, what is the length of the shortest path between A and C?



Originally Problem 10 from Fillins round of 2016 Georgia Tech High School Mathematics Competition.

We received 11 submissions. We present the solution by Carlos Moreno and Ángel Plaza.

Imagine flattening two triangular faces of the pyramid as shown.



Since $EB = 50$, let $x + y = 50$ as shown. By the Pythagorean theorem, it is deduced $x = 36$, $y = 14$. Then the height over EB in triangle ABC is $h = 48$ and consequently the shortest path between A and C is $2 \times 48 = 96$.

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CC314. An infinite sequence $a_1, a_2, \ldots$ of 1's and 2's is uniquely defined by the following properties:

1. $a_1 = 1$ and $a_2 = 2$,
2. For every $n \geq 1$, the number of 1's between the nth 2 and the $(n+1)^{\text{st}}$ 2 is equal to a_{n+1}.

Is the sequence periodic from the beginning?

Originally Problem 4 from Proof round of 2016 Georgia Tech High School Mathematics Competition.

We received 4 submissions of which 2 were both correct and complete. Both correct solutions used proof-by-contradiction strategy. We present the solution by Steven Chow.

Assume for the sake of contradiction that the sequence is periodic from the beginning. Let k be the fundamental period of the sequence. Let t be the number of 2's in $a_1, a_2, \ldots, a_k$. Due to periodicity, t is the number of 2's in any subsequence of length k.

We claim that for all integers $j \geq 2$, $a_j = a_{j+t}$. Indeed a_j is the number of 1's between the $(j-1)^{\text{st}}$ 2 and the j^{th} 2, and a_{j+t} is the number of 1's between the $(j+t-1)^{\text{st}}$ 2 and the $(j+t)^{\text{th}}$ 2. Between the $(j-1)^{\text{st}}$ 2 and the $(j+t-1)^{\text{st}}$ 2, including the $(j-1)^{\text{st}}$ 2 and excluding the $(j+t-1)^{\text{st}}$ 2, there are exactly t 2's. The sequence is periodic with exactly t 2's in any subsequence of length equal to the fundamental period. Therefore the subsequence starting at the $(j+t-1)^{\text{st}}$ 2 must be a repeat of the subsequence starting at the $(j-1)^{\text{st}}$ 2. Therefore $a_j = a_{j+t}$.

Since $a_j = a_{j+t}$ for all integers $j \geq 2$, the sequence $a_2, a_3, \ldots$ is periodic with period t. However, the fundamental period is k, so $k \leq t$. From the definition of t, $t \leq k$. Therefore $t = k$ and the sequence consists of 2's only.

This is a contradiction. Therefore, the sequence is not periodic from the beginning.

Editor's Comment. The above solution can be slightly modified to obtain that the sequence is not periodic starting at n, for any $n \geq 1$.

CC315. A square table is divided into a 3×3 grid with every cell having 3 coins. In every step of a game, Terry can take 2 coins from the table as long as they come from distinct but adjacent cells. (Here "adjacent" means the two cells share a common edge.) At most how many coins can Terry take?

Originally Problem 9 from Fillins round of 2016 Georgia Tech High School Mathematics Competition.

We received 7 submissions. We present the solution by the Missouri State University Problem Solving Group.

The answer is 24 coins.

If we apply the usual checkerboard colouring with the central and corner cells coloured black and the side squares coloured white, there are 15 coins in black cells and 12 coins in white ones. But every time a pair of coins is chosen, one must be from a black cell and one from a white one. At best this will leave us with 3 coins remaining (i.e. 24 coins taken).

This result can actually be obtained by choosing pairs of coins from adjacent cells on the left-hand side of the first three rows and from two adjacent cells in the last column.

A similar argument shows that if k coins are placed in each cell of an $m \times n$ grid with m and n both odd, then at most $k(mn-1)$ coins can be taken.

Editor's Comments. There was quite a variety of solutions here. Many did *one* of the following:

- Using a checkerboard argument, showed that it is impossible to take more than 24 coins.
- By construction, showed that it is possible to take exactly 24 coins.

Only a few showed both sides of the proof.

CC316. Positive integers (x, y, z) form a *Trenti-triple* if $3x = 5y = 2z$. Show that for every Trenti-triple (x, y, z) the product xyz must be divisible by 900.

Originally Question 3 from Hypatia 2011.

We received 13 submissions, all correct. We feature a composite solution based on essentially the same ones given by most of these solvers.

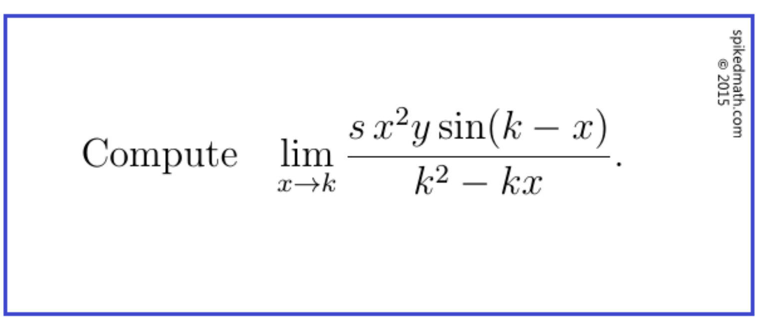
Note first that 3, 5, and 2 are mutually relatively co-prime.

Hence, from $3x = 5y = 2z$ we deduce that $5 \mid x$ and $2 \mid x$, so $10 \mid x$.

Similarly, from $5y = 2z = 3x$ we deduce that $2 \mid y$ and $3 \mid y$, so $6 \mid y$.

Finally, from $2z = 3x = 5y$ we deduce that $3 \mid z$ and $5 \mid z$, so $15 \mid z$.

It then follows that xyz is divisible by $10 \cdot 6 \cdot 15$ or 900.



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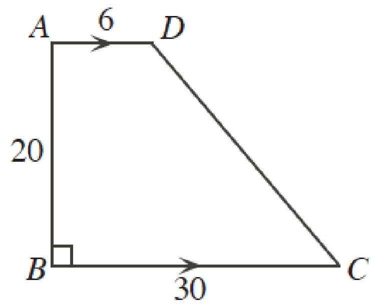
CC317. In the diagram, $ABCD$ is a trapezoid with AD parallel to BC and BC perpendicular to AB. Also, $AD = 6$, $AB = 20$, and $BC = 30$.



1. Determine the area of trapezoid $ABCD$.
2. There is a point K on AB such that the area of triangle KBC equals the area of quadrilateral $KADC$. Determine the length of BK.
3. There is a point M on DC such that the area of triangle MBC equals the area of quadrilateral $MBAD$. Determine the length of MC.

Originally Question 3 from Hypatia 2008.

We received 8 correct solutions. We present the solution by Ricard Peiró i Estruch.

1. The area of $ABCD$ is $\frac{30+6}{2} \cdot 20 = 360$.

2. Set $x = BK$. The area of triangle KBC is half the area of trapezoid $ABCD$, so

$$\frac{1}{2} \cdot 30x = \frac{1}{2} \cdot 360.$$

Solving gives $x = 12$.

3. Let D' be the projection of D onto line BC. Then $DD' = AB = 20$ and

$$D'C = BC - AD = 24.$$

Applying the Pythagorean Theorem to right triangle $DD'C$ gives

$$CD = \sqrt{24^2 + 20^2} = 4 \cdot \sqrt{61}.$$

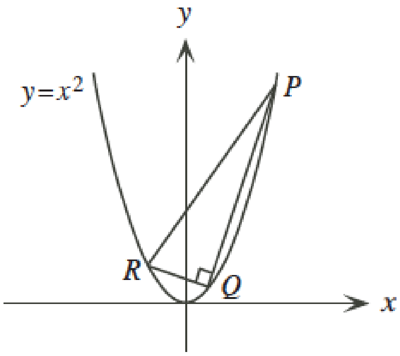
Since the area of triangle MBC is half the area of trapezoid $ABCD$ and thus equal to the area of triangle KBC, and the two triangles share base $\overline{BC}$, they have the same height. Thus $\overline{KM}$ is parallel to both $\overline{BC}$ and $\overline{AD}$. It follows that

$$\frac{MC}{BK} = \frac{CD}{AB} \quad \text{or} \quad \frac{MC}{12} = \frac{4\sqrt{61}}{20}.$$

Solving gives

$$MC = \frac{12\sqrt{61}}{5}.$$

CC318. Right-angled triangle PQR is inscribed in the parabola with equation $y = x^2$, as shown. Points P, Q and R have coordinates $(p, p^2), (q, q^2)$ and (r, r^2), respectively. If p, q and r are integers, show that $2q + p + r = 0$.



Originally Question 3c from Hypatia 2012.

We received 9 submissions of which all but one were correct and complete. We present the solution by Digby Smith.

Note from the diagram that $p > q > r$. Since $\angle RQP = 90^\circ$ it follows that QR and QP are perpendicular. Moreover, the slope of line QP is

$$m_{QP} = (p^2 - q^2)/(p - q) = p + q,$$

the slope of line QR is

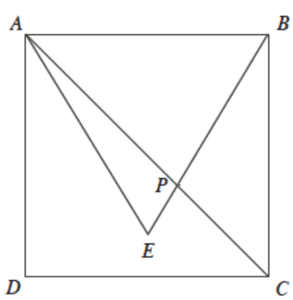
$$m_{QR} = (q^2 - r^2)/(q - r) = q + r,$$

and

$$m_{QP}m_{QR} = (p + q)(q + r) = -1.$$

Since p, q, and r are integers it then follows that either $p + q = 1$ and $q + r = -1$, or $p + q = -1$ and $q + r = 1$. In both cases, after adding the two equations, it follows that $2q + p + r = 0$.

CC319. In the diagram, square $ABCD$ has sides of length 4, and triangle ABE is equilateral. Line segments BE and AC intersect at P. Determine the exact area of triangle APE.



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Originally Question 3 from Hypatia 2010.

We received 14 correct solutions from 11 solvers. The solver of one incorrect submission apparently misread the question thinking that the side length of the given square is 1. We present the solution by Richard Peiró i Estruch.

From P, draw line PH perpendicular to AB and let $\overline{AH} = x$.

Since $\angle BAP = 45°$, we have $\overline{PH} = \overline{AH} = x$.

Since
$$\overline{PH} = \overline{(BH)} \tan 60° = (4 - x)\sqrt{3},$$
we have
$$x = \sqrt{3}(4 - x).$$
Solving yields
$$x = \frac{4\sqrt{3}}{\sqrt{3}+1} = 2\sqrt{3}(\sqrt{3} - 1) = 6 - 2\sqrt{3}.$$
Finally,
$$\begin{aligned} |\triangle APE| &= |\triangle ABE| - |\triangle ABP| \\ &= \frac{1}{2}(4)(2\sqrt{3}) - \frac{1}{2}(4)(6 - 2\sqrt{3}) = 4\sqrt{3} - 4(3 - \sqrt{3}) \\ &= 8\sqrt{3} - 12. \end{aligned}$$

CC320. A sequence of m P's and n Q's with $m > n$ is called *non-predictive* if there is some point in the sequence where the number of Q's counted from the left is greater than or equal to the number of P's counted from the left. For example, if $m = 5$ and $n = 2$, the sequences $PPQQPPP$ and $QPPPQPP$ are non-predictive.

1. If $m = 7$ and $n = 2$, determine the number of non-predictive sequences that begin with P.
2. Suppose that $n = 2$. Show that for every $m > 2$, the number of non-predictive sequences that begin with P is equal to the number of non-predictive sequences that begin with Q.
3. Determine the number of non-predictive sequences with $m = 10$ and $n = 3$.

Originally Question 3 from Hypatia 2013.

We received five submissions. We present the solution by Kathleen E. Lewis.

1. The sequence must begin with PP or PQ. If it begins with PP and has only 2 Q's, then the only way that it can be non-predictive is to begin $PPQQ$. But then the rest of the terms must be P's, so the only such sequence is $PPQQPPPPP$. Now suppose that it begins with PQ. Then it's already non-predictive, so the

remaining terms can be in any order. Since there are 6 P's and one Q remaining, there are 7 possible arrangements. Therefore, there are a total of eight non-predictive sequences beginning with P.

2. Any sequence that begins with Q is automatically non-predictive. There are as many such sequences as there are arrangements of the other letters, one Q and m P's. Since there are $m+1$ places to put the Q, there are $m+1$ such sequences. The number of non-predictive sequences beginning with P can be analyzed as in part (1) above. There is only one such sequence beginning with PP, since it must start with $PPQQ$. Any sequence beginning with PQ is non-predictive, so we can arrange the other terms in any way. There are $m-1$ P's and only one Q remaining, so there are m ways to arrange them. That gives a total of $m+1$ non-predictive sequences beginning with P to match the $m+1$ beginning with Q.

3. Consider the following cases:

a) sequences starting with Q: Any sequence starting with Q is automatically non-predictive. The remaining 10 P's and 2 Q's can be arranged in any order, so there are $\binom{12}{2} = 66$ such sequences.

b) sequences starting with PQ: Again, all such sequences are non-predictive. There are 9 P's and 2 Q's still to arrange, so that gives $\binom{11}{2} = 55$ such sequences.

c) sequences starting with PPP: The only way such a sequence can be non-predictive is to have the three Q's immediately follow the first three P's, so the only non-predictive sequence of this form is $PPPQQQPPPPPPP$.

d) sequences starting with $PPQQ$: Any such sequence is non-predictive. Since there is one remaining Q to place among 8 P's, there are nine such sequences.

e) sequences starting with $PPQP$: The only way such a sequence can be non-predictive is to continue as $PPQPQQPPPPPPP$. So there is only one sequence in this case.

Total: Hence, there is a total of $66+55+1+9+1 = 132$ non-predictive sequences with 10 P's and 3 Q's.

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