

PROBLEM SOLVING VIGNETTES

No.2

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Carefully Counting Canoeists

This month, we will look at problem **7a** from the 2018 *Euclid Contest*, hosted by the The Centre for Education in Mathematics and Computing at the University of Waterloo. You can check out the contest, and past contests on the CEMC website at www.cemc.uwaterloo.ca.

Eight people, including triplets Barry, Carrie and Mary, are going for a trip in four canoes. Each canoe seats two people. The eight people are to be randomly assigned to the four canoes in pairs. What is the probability that no two of Barry, Carrie and Mary will be in the same canoe?

This problem can be tricky as it involves careful counting. Since it is a probability problem we may also interpret the process of putting people into canoes in different ways, as long as we are consistent throughout our solution. We need to compute the number of ways that the people can be seated, without restriction, and the number of ways that they can be seated with the triplets separated. We will look at the problem in several different ways.

Solution #1: Completely ordered.

We will impose an “order” on the canoes and on the seats in the canoes. That is, we will consider Alice and Barry in the red canoe as different from Alice and Barry in the green canoe. Similarly, we will consider Alice and Barry in the silver canoe with Alice in front different from Alice and Barry in the silver canoe with Barry in the front. Thus there are 8 positions into which we want to order our 8 people. We can do the ordering in $8!$ ways.

To ensure the triplets are in separate canoes, we will assign them separately. Barry can be placed in any of the 8 seats. When it comes to seating Carrie, she cannot be in the same seat as Barry or the other empty seat in his canoe. As such, there are 6 possible seats in which to seat Carrie. Similarly, there are 4 possible seat choices for Mary. Once the triplets have been seated, we can seat the remaining 5 people in any order, which can be done in $5!$ ways. Thus the total number of ways to seat the people, keeping the triplets separated is $8 \times 6 \times 4 \times 5!$.

Hence, the desired probability is

$$\frac{8 \times 6 \times 4 \times 5!}{8!} = \frac{8 \times 6! \times 4}{8 \times 6! \times 7} = \frac{4}{7}.$$

Solution #2: Partially ordered.

We will impose an order on the canoes, but not on the positions within the canoes. Thus we need to pick 2 people for the first canoe, which we can do in $\binom{8}{2}$ ways. Filling the other canoes in similar ways, we get the total number of ways to fill the canoes to be $\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$. We could have similarly assumed that the 8 people were given cards randomly that had either R , G , B , or S on it, indicating that the canoeist was assigned to the red, green, blue or silver canoe, respectively. We can then think of the process as the number of ways of ordering two each of R , G , B , and S . This can be done in $\frac{8!}{(2!)^4}$ ways. It is easy to show that

$$\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{8!}{(2!)^4}$$

Next, since we want the triplets separated, we will deal with them first. First choose which canoes they will be placed into, which can be done in $\binom{4}{3}$ ways. Since we have “ordered” our canoes, the order the triplets are placed in the canoes is important. They can be placed in the chosen canoes in $3!$ ways. At this point we have one empty canoe, and three half filled canoes in which to place our remaining 5 canoeists. We will fill the empty canoe first, which can be done in $\binom{5}{2}$ ways. Then the remaining 3 canoeists have to be “ordered” into the three remaining canoes with the triplets, which can be done in $3!$ ways. Note we could have put the three canoeists with the triplets first and then filled the empty canoe with the remaining two. You may wish to show that this yields the same result. So the total number of ways to seat the people, keeping the triplets separated is $\binom{4}{3} \times (3!)^2 \times \binom{5}{2}$.

Once again, our desired probability is

$$\frac{\binom{4}{3} \times (3!)^2 \times \binom{5}{2}}{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}} = \frac{4}{7}.$$

Solution #3: Completely unordered.

In this solution we will not impose an order on either the canoes or the positions. From solution #2, the number of ways to place the canoeists in the boats where the boats are “ordered” but the seats are not is $\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$. If we want the boats not to be ordered, we need to “remove” the order of the four boats by noticing that each possible set of groupings of pairs can be assigned to the “ordered” canoes in $4!$ ways. Thus the number of unordered assignments is $\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{4!}$.

To assign the pairs with the triplets separated, we can assume that each triplet, in turn, will select a person with whom they will be paired. This can be done in $5 \times 4 \times 3$ ways. The remaining two people form the final pair. Thus the desired probability is

$$\frac{5 \times 4 \times 3}{\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{4!}} = \frac{4}{7}$$

Solving a problem in different ways gives you different insights. Solution #3 has the most straightforward way of calculating the number of ways to seat the canoeists with the triplets separated. It also seems like the most “real world” way of thinking of the problem (i.e. canoeists probably were assigned a partner, not a particular canoe or seating order). On the other hand, determining the number of ways to seat the canoeists without any order, becomes a little trickier. Solution #2 seems like a natural way to think about the problem, and was the way I first thought of it. Even if we are not assigning groups to particular boats, if we pick the groups by picking names out of a hat, the *process* imposes an order on the groups, even if that order is removed later on. Solution #1, while seemingly unnatural, does provide the “cleanest” solution. When dealing with probability questions it is worth thinking about considering the problems as ordered, or not. In some cases, going against the “natural” way of thinking about the problem might provide you with a simplified solution. It is worth your time to check out the official solutions to the problem from the CEMC website. Their first solution is similar to our solution #2 with a few interesting variations. Their second solution is a clever way to determine the probability without counting the groups.

Careful counting is important when dealing with probability or counting problems. I suspect that many students who did the problem, but got it wrong, probably computed the two totals for the problem with different interpretations. I wouldn't be surprised if there were many solutions that gave

$$\frac{5 \times 4 \times 3}{\binom{8}{2} \binom{6}{2} \binom{4}{2} \binom{2}{2}},$$

where students were thinking that they were dealing with an unordered problem but, inadvertently, imposed an ordering to the canoes. We will revisit counting problems in future columns.

