SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2017: 43(7), p. 302–306.

4261. *Proposed by Margarita Maksakova.*

Consider the chess board. A baron can move only on the black squares and in one move he can go from one black square to any of the diagonally adjacent black squares. What is the smallest number of moves he needs to go to every black square?

We received no correct and complete solution, so the problem remains open. One submission showed that it is possible to go through every square in 34 moves. Can you show whether this is the smallest number of moves needed?

4262. *Proposed by Prithwijit De.*

Let $a_1, a_2, \dots, a_n$ be positive integers and suppose $\sum\limits_{k=1}^{n} a_k = S$. Find the smallest positive value of c such that the equation

$$\sum_{k=1}^{n} \frac{a_k x^k}{1+x^{2k}} = c$$

has a unique real solution.

We received 6 correct solutions and one incomplete submission. We present the solution of Ivko Dimitrić, slightly modified by the editor.

Let $L(x)$ denote the left hand side of the equation. Then $L(0) = 0$. Suppose $x \neq 0$. Then

$$L(x) = \sum_{k=1}^{n} \frac{a_k x^k}{1+x^{2k}} = \sum_{k=1}^{n} \frac{a_k}{x^k + \frac{1}{x^k}}.$$

Note that $L(x) = L(1/x)$. We have

$$\begin{aligned} L(x) = \sum_{k=1}^{n} \frac{a_k}{x^k + \frac{1}{x^k}} &\leq \frac{1}{2}\sum_{k=1}^{n} a_k \frac{2}{|x^k + \frac{1}{x^k}|} = \frac{1}{2}\sum_{k=1}^{n} a_k \frac{2}{|x|^k + \frac{1}{|x|^k}} \\ &\leq \frac{1}{2}\sum_{k=1}^{n} a_k \sqrt{|x|^k \cdot \frac{1}{|x|^k}} = \frac{1}{2}\sum_{k=1}^{n} a_k = \frac{S}{2}, \end{aligned}$$

where the last inequality is the AM-GM inequality. Note that equality $L(x) = \frac{S}{2}$ is satisfied if and only if $x > 0$ and $x^k = 1/x^k$ for all k, implying $x = 1$ is a unique solution for $L(x) = S/2$. Now suppose $0 < c < S/2$. By the intermediate value theorem and $L(0) = 0$, there exists $0 < r < 1$ with $L(r) = c$. But since $L(1/r) = L(r)$, the equation $L(x) = c$ does not have a unique solution. Thus $c = S/2$.

4263. *Proposed by Michel Bataille.*

Let ABC be a triangle. Let Γ, with centre O and radius R, be the circumcircle of ABC and γ, with centre $I \neq O$ and radius r, be the incircle of ABC. Let D, E, F be the orthogonal projections of the inverse of I in Γ onto BC, CA, AB, respectively. Express the circumradius of ΔDEF as a function of R and r.

We received 3 submissions, all of which were correct; we feature the solution by Andrew David Ionascu, slightly modified by the editor.

We denote the inverse of I with respect to Γ by J (that is, $OJ \times OI = R^2$), and denote by D', E', and F' the points of tangency where the incircle touches the respective sides BC, CA, and AB. Note that because $\frac{OJ}{OA} = \frac{OA}{OI}$, the triangles JAO and AIO are similar by side-angle-side. Therefore, $\frac{JA}{AI} = \frac{AO}{IO} = \frac{R}{IO}$, which by Euler's formula (namely $OI^2 = R^2 - 2Rr$) becomes

$$\frac{JA}{AI} = \frac{R}{\sqrt{R^2 - 2Rr}}.$$

Because of the right angles at E and F, the circle whose diameter is JA contains E and F and is therefore the circumcircle of ΔEAF. The Law of Sines applied to this triangle gives us

$$\frac{EF}{\sin \angle EAF} = JA. \tag{1}$$

Similarly, because of right angles at E' and F', the circle whose diameter is IA contains E' and F' and is therefore the circumcircle of $\Delta E'AF'$, whence

$$\frac{E'F'}{\sin \angle E'AF'} = IA. \tag{2}$$

Because $E' \in AE$ and $F' \in AF$, the angles $\angle EAF$ and $E'AF'$ are equal or supplementary, and division using equations (1) and (2) yields

$$\frac{EF}{E'F'} = \frac{JA}{AI} = \frac{R}{\sqrt{R^2 - 2Rr}}.$$

In the same way, we can show that

$$\frac{FD}{F'D'} = \frac{DE}{D'E'} = \frac{R}{\sqrt{R^2 - 2Rr}}.$$

Thus triangles DEF and $D'E'F'$ are similar (by side-side-side), so that if x is the circumradius of ΔDEF (which we seek) while the inradius r is the circumradius of $\Delta D'E'F'$, we must also have $\frac{x}{r} = \frac{R}{\sqrt{R^2-2Rr}}$. We conclude that the circumradius of ΔDEF satisfies

$$x = \frac{Rr}{\sqrt{R^2-2Rr}}.$$

Editor's comments. The proposer, whose key step was the same as that of our featured solution, observed that the result can be found as a theorem in Nathan Altshiller-Court, *College Geometry*, Dover, 1980, paragraph 362, p. 173: *The pedal triangles of two points for a given triangle are similar if and only if the two points are inverse with respect to the circumcircle of the given triangle.*

4264. *Proposed by Dorin Marghidanu and Leonard Giugiuc.*

Let (a_n) and (b_n) be two sequences such that $a_0, b_0 > 0$ and

$$a_{n+1} = a_n + \frac{1}{2b_n} \quad \text{and} \quad b_{n+1} = b_n + \frac{1}{2a_n}$$

for all $n \geq 0$. Prove that

$$\max(a_{2017}, b_{2017}) > 44.$$

We received 11 correct solutions. We present here the solution by Paolo Perfetti.

The quantity

$$f(a_n, b_n) = \frac{a_n}{b_n}$$

is invariant under the recurrence. Indeed,

$$f(a_{n+1}, b_{n+1}) = \frac{a_n + \frac{1}{2b_n}}{b_n + \frac{1}{2a_n}} = \frac{a_n}{b_n} = f(a_n, b_n)$$

This means that

$$\frac{a_n}{b_n} = f(a_n, b_n) = f(a_0, b_0) = \frac{a_0}{b_0}$$

and then

$$a_{n+1} = a_n + \frac{1}{2a_n}\frac{a_0}{b_0} \quad \text{and} \quad b_{n+1} = b_n + \frac{1}{2b_n}\frac{b_0}{a_0}.$$

Thus,

$$a_{n+1}^2 + b_{n+1}^2 = a_n^2 + b_n^2 + \underbrace{\left(\frac{a_0}{b_0} + \frac{b_0}{a_0}\right)}_{\geq 2\,(AGM)} + \underbrace{\left(\frac{1}{4a_n^2}\frac{a_0^2}{b_0^2} + \frac{1}{4b_n^2}\frac{b_0^2}{a_0^2}\right)}_{\geq 0} \geq \; a_n^2 + b_n^2 + 2,$$

and

$$a_{n+1}^2 + b_{n+1}^2 > 2(n+1) + a_0^2 + b_0^2 > 2(n+1).$$

Moreover,

$$\max(a_{2017}, b_{2017}) \geq \frac{\sqrt{a_{2017}^2 + b_{2017}^2}}{\sqrt{2}} > \sqrt{2017} > 44.9.$$

4265. *Proposed by Daniel Sitaru.*

Consider real numbers $a, b, c \in (0, 1)$ such that $a + b + c = 1$. Show that

$$\frac{4}{\pi}(\arctan a + \arctan b + \arctan c) > \frac{1}{2 - (ab + bc + ca)}.$$

There were 9 correct solutions and 2 incorrect submissions submitted. We follow the independent solutions of Daniel Dan; and the team D. Bailey, E. Campbell, and C. Diminnie.

Since $(4/\pi) \arctan x$ is concave for $x \geq 0$ and is equal to x for $x = 0$ and $x = 1$,

$$\frac{4}{\pi} \arctan x \geq x$$

for $0 \leq x \leq 1$. Therefore the left side of the inequality is not less than $a+b+c = 1$.

Since

$$\begin{aligned} 2(ab + bc + ca) &= (a + b + c)^2 - (a^2 + b^2 + c^2) \\ &= 1 - (a^2 + b^2 + c^2) \\ &\leq 1 - (ab + bc + ca), \end{aligned}$$

then $ab + bc + ca \leq 1/3$ and

$$\frac{1}{2 - (ab + bc + ca)} \leq \frac{3}{5} < 1.$$

The result follows.

Editor's comments. Some solvers used $\arctan x \leq x-(x^3/3)$ and standard inequalities to get the lower bound $8/3\pi$ for the left side. One solver used Karamata's inequality for the concave function $\arctan x$ and the triples (a, b, c), $(1, 0, 0)$ to show that the left side was not less than $4/\pi$.

4266. *Proposed by Marius Stănean.*

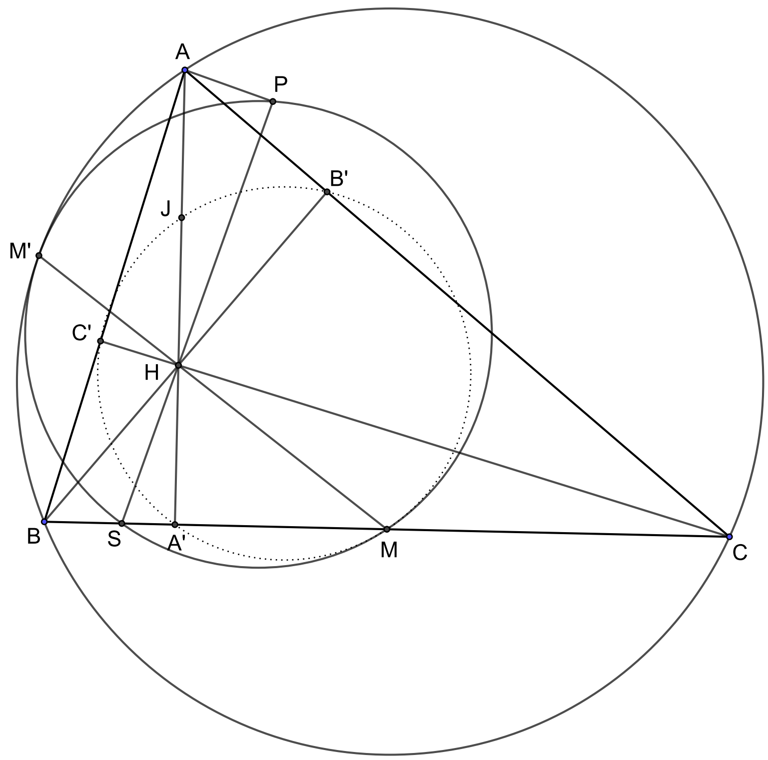
Let ABC be a triangle with orthocenter H. Let HM be the median and HS be the symmedian in triangle BHC. Denote by P the orthogonal projection of A onto HS. Prove that the circumcircle of triangle MPS is tangent to the circumcircle of triangle ABC.

We received 3 solutions. We present the solution by Michel Bataille modified by the editor.

To ensure the existence of $\triangle BHC$, we assume that ΔABC is not right-angled at B or C. Consider the case when $\triangle ABC$ is right-angled at A; then A, H and P coincide. The circumcentre of $\triangle ABC$ is M. Moreover, one can show that S coincides with the foot of the altitude from A, hence the circumcentre of

$\triangle PSM = \triangle ASM$ is the midpoint of AM. The required result follows in this case.

From now on, we suppose that ΔABC is not right-angled. Below is included a diagram for the case when $\triangle ABC$ is acute (the argument will also work when $\triangle ABC$ is obtuse, but points might be in a different order on lines and circles).



Let A', B', C' be the feet of the altitudes of $\triangle ABC$ (with A' on BC, and so on), and J be the midpoint of the line segment AH. Denote by $\mathbf{I}$ the inversion with centre H such that $\mathbf{I}(B) = B'$. Consider the effect of this inversion on the two circles in which we are interested.

Look at $\bigcirc ABC$, the circumcircle of $\triangle ABC$. Since $\angle BB'C = \angle CC'B = 90°$, the points B, B', C and C' are concyclic. The power of the point H gives us $HB' \cdot HB = HC' \cdot HC$, whence $\mathbf{I}(C) = C'$. Similarly, $\mathbf{I}(A) = A'$. Hence the image of the circumcircle of $\triangle ABC$ under $\mathbf{I}$ is the Euler circle $\bigcirc A'B'C'$ (a.k.a. the 9-point circle, which is known to also go through M and J, and which is shown dotted in the diagram).

Now consider the circumcircle of $\triangle PSM$. Similar to the above, we have $\angle APS = \angle AA'S = 90°$, so the points A, A', P and S are concyclic and the power of the point H gives us $HA' \cdot HA = HP \cdot HS$. Recall that $\mathbf{I}(A) = A'$, so it follows that $\mathbf{I}(P) = S$. Since the two points P and S on $\bigcirc PSM$ get mapped to each other, we conclude that the image of $\bigcirc PSM$ is itself.

It follows that, in order to prove that $\bigcirc ABC$ and $\bigcirc PSM$ are tangent, it suffices to show that $\bigcirc A'B'C'$ and $\bigcirc PSM$ are tangent. To this end, we will show that the centre of the Euler circle, the centre of $\bigcirc PSM$, and the point M are collinear. Denote $\mathbf{I}(M)$ by M'; note that M' is on both $\bigcirc ABC$ and $\bigcirc PSM$.

From earlier, $\mathbf{I}(S) = P$ and $\mathbf{I}(A') = A$; it follows that the line through SA' gets mapped to the circumcircle of $\triangle PAH$ (recall that an inversion with center H will map lines to circles that go through H). Thus M', B' and C', which are images of points on the line segment BC, are all on $\bigcirc PAH$. From the given setup of the problem, $\angle BHS = \angle CHM$, so $\angle M'HC' = \angle PHB'$; thus the arcs $M'C'$ and PB' on $\bigcirc PAH$ are congruent, which implies $M'P \parallel C'B'$. Denote by l the perpendicular bisector of $M'P$, which is necessarily also the perpendicular bisector of $C'B'$.

$M'P$ is a chord in $\bigcirc PSM$, so the centre of $\bigcirc PSM$ is on l. $C'B'$ is a chord in $\bigcirc A'B'C'$, so the centre of the Euler circle is on l. Finally, note that $\triangle MB'C'$ is isosceles, since M is the midpoint of the hypotenuse of $\triangle BB'C$, implying $MB' = MC$, and also the midpoint of the hypotenuse of $\triangle CC'B$, implying $MC' = MC$. Hence M is also on the perpendicular bisector l of $B'C'$. This concludes the argument that $\bigcirc PSM$ and the Euler circle are tangent at M, and applying the inversion $\mathbf{I}$ gives us that $\bigcirc PSM$ and $\bigcirc ABC$ are tangent at M'.

4267. *Proposed by Leonard Giugiuc.*

Let a, b, c and d be real numbers such that $0 < a, b, c \leq 1$ and $abcd = 1$. Prove that

$$5(a+b+c+d) + \frac{4}{abc+abd+acd+bcd} \geq 21.$$

There were 7 correct solutions and two incorrect submissions, as well as one that made use of Maple. *Some of the solutions were quite complicated. We present the solution by Kee-Wai Lau and Angel Plaza, done independently.*

We first note that $d = 1/abc$ and

$$\begin{aligned}&(a+b+c+d) - (abc+abd+acd+bcd)\\ &= d(a^2bc + ab^2c + abc^2 + 1 - a^2b^2c^2 - ab - ac - bc)\\ &= d(1-ab)(1-ac)(1-bc) \geq 0,\end{aligned}$$

so that

$$\frac{a+b+c+d}{abc+abd+acd+bcd} \geq 1.$$

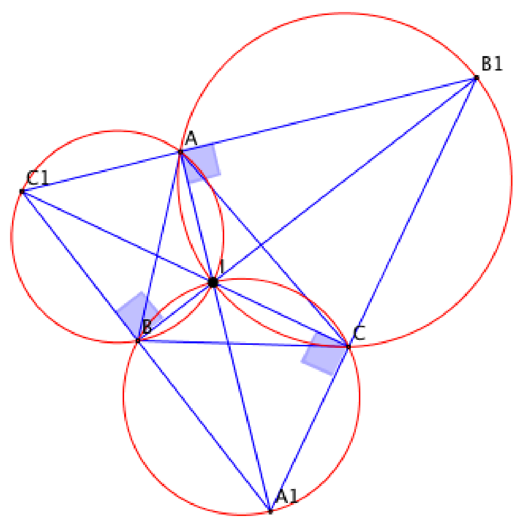
Therefore, applying the arithmetic-geometric means inequality twice, we find that

$$\begin{aligned}
&5(a+b+c+d)+\frac{4}{abc+abd+acd+bcd}\\
&=19\left(\frac{a+b+c+d}{4}\right)+\left[\frac{a+b+c+d}{4}+\frac{4}{abc+abd+acd+bcd}\right]\\
&\geq 19(abcd)^{1/4}+2\left[\frac{a+b+c+d}{abc+abd+acd+bcd}\right]^{1/2}\geq 19+2=21.
\end{aligned}$$

4268. *Proposed by Mihaela Berindeanu.*

Let I be the incenter of the acute triangle ABC, and let the triangle's internal angle bisectors intersect the circles IBC, ICA, and IAB again at A_1, B_1, and C_1, respectively. Show that $\overrightarrow{IA_1}+\overrightarrow{IB_1}+\overrightarrow{IC_1}=\overrightarrow{0}$ if and only if $\triangle\, ABC$ is equilateral.

We received 14 submissions, all of which were correct; we feature a composite of similar solutions by Ivko Dimitrić and Titu Zvonaru.



The exterior angle at I of $\triangle AIC$ satisfies $\angle A_1IC=\frac{A+C}{2}$. Since I, B, A_1, C are concyclic (in that order), we have $\angle CA_1I=\angle CBI=B/2$. Hence,

$$\angle ICA_1=180^\circ-\frac{A+C}{2}-\frac{B}{2}=90^\circ$$

and in the same manner, $\angle B_1CI=90^\circ$, so that $\angle B_1CA_1=180^\circ$, which means that the points A_1, C, and B_1 are collinear with $C_1C\perp A_1B_1$. Analogously, B_1, A, C_1 are collinear with $A_1A\perp B_1C_1$, and C_1, B, A_1 are collinear with $B_1B\perp C_1A_1$. Hence, A_1A, B_1B and C_1C are the altitudes of the triangle $A_1B_1C_1$ so that the incenter I of the given triangle ABC is the orthocenter of $\triangle A_1B_1C_1$.

Since for any point I in the plane of triangle $A_1B_1C_1$ one has

$$\overrightarrow{IA_1} + \overrightarrow{IB_1} + \overrightarrow{IC_1} = 3\overrightarrow{IG_1},$$

where G_1 is the centroid of $\triangle A_1B_1C_1$, this sum will be zero if and only if I is the centroid of the said triangle. But, the centroid and incenter of a triangle coincide if and only if the triangle is equilateral. The problem has therefore been reduced to proving that ΔABC is equilateral if and only if $\Delta A_1B_1C_1$ is equilateral. We have ΔABC is equilateral if and only if $\angle AIB = \angle BIC = \angle CIA = 120°$, if and only if $\angle BC_1A = \angle CA_1B = \angle AB_1C = 60°$, if and only if $\Delta A_1B_1C_1$ is equilateral.

Editor's comments.

(1) Only Leonard Giugiuc observed explicitly that there is no need to require that ΔABC be acute (as our featured solution shows).

(2) It is a standard result that the vertices of $\Delta A_1B_1C_1$ are the excenters of ΔABC (see, for example, Chapter X of Roger A. Johnson, *Advanced Euclidean Geometry*), and many of the submissions made use of well-known properties of this pair of triangles to shorten their arguments.

(3) Anna Valkova Tomova used an argument much like our featured solution to extend the result to

> *ΔABC is isosceles with apex at A if and only if there exists a nonzero real number λ for which $\lambda\overrightarrow{IA_1} + \overrightarrow{IB_1} + \overrightarrow{IC_1} = 0$.*

4269. *Proposed by Hung Nguyen Viet.*

Let $x_1, x_2, \ldots, x_n$ be real numbers such that

$$\sin x_1 \cos x_2 + \sin x_2 \cos x_3 + \cdots + \sin x_n \cos x_1 = \frac{n}{2}.$$

Prove that

$$\cos 2x_1 + \cos 2x_2 + \cdots + \cos 2x_n = 0.$$

There were 15 correct solutions submitted, 9 of which had essentially the argument given below. The remainder relied on an inequality forced to equality by the same upper and lower bounds.

With $x_{n+1} = x_1$, we have that

$$\begin{aligned}\sum_{k=1}^{n}(\sin x_k - \cos x_{k+1})^2 &= \sum_{k=1}^{n}(\sin^2 x_k + \cos^2 x_{k+1}) - 2\sum_{k=1}^{n}\sin x_k \cos x_{k+1} \\ &= \sum_{k=1}^{n}(\sin^2 x_k + \cos^2 x_k) - 2(n/2) = n - n = 0,\end{aligned}$$

so that $\sin x_k = \cos x_{k+1}$ for each k. Therefore,

$$\sum_{k=1}^{n}\cos 2x_k = \sum_{k=1}^{n}(\cos^2 x_k - \sin^2 x_k) = \sum_{k=1}^{n}(\cos^2 x_{k+1} - \sin^2 x_k) = 0.$$

4270. *Proposed by Leonard Giugiuc.*

Let k and t be real numbers with $k \in (0,1)$ and $t \in [\frac{\pi}{4}, \frac{\pi}{2}]$. Prove that

$$\int_0^t \frac{\cos x}{x^k} dx \geq \int_0^t \frac{\sin x}{x^k} dx.$$

We received 10 solutions and will feature just one of them here, by Michel Bataille.

Let $t \in [\frac{\pi}{4}, \frac{\pi}{2}]$. For $x \in (0, t]$, we have

$$0 \leq \frac{\cos x}{x^k} \leq \frac{1}{x^k}$$

and $\int_0^t \frac{1}{x^k}\, dx$ exists, hence the integral $\int_0^t \frac{\cos x}{x^k}\, dx$ exists. The integral $\int_0^t \frac{\sin x}{x^k}\, dx$ also exists since

$$\lim_{x \to 0^+} \frac{\sin x}{x^k} = \lim_{x \to 0^+} x^{1-k} \cdot \frac{\sin x}{x} = 0 \cdot 1 = 0.$$

Now, let

$$I = \int_0^{\pi/4} \frac{\cos x - \sin x}{x^k}\, dx, \quad F(t) = \int_{\pi/4}^t \frac{\cos x - \sin x}{x^k}\, dx, \quad G(t) = I + F(t).$$

We are required to prove that $G(t) \geq 0$.

Since

$$x \mapsto \frac{\cos x - \sin x}{x^k}$$

is continuous on $[\frac{\pi}{4}, \frac{\pi}{2}]$, the function F is differentiable on this interval and so is the function G with $G'(t) = F'(t) = \frac{\cos t - \sin t}{t^k}$. For $t \in (\frac{\pi}{4}, \frac{\pi}{2}]$, we have $\cos t < \sin t$, hence $G'(t) < 0$ and therefore G is decreasing on $[\frac{\pi}{4}, \frac{\pi}{2}]$. As a result, it is sufficient to show that $G(\pi/2) \geq 0$. To this aim, we consider

$$\frac{\sqrt{2}}{2} \cdot G(\pi/2) = \int_0^{\pi/4} \frac{\sin(\frac{\pi}{4} - x)}{x^k}\, dx + \int_{\pi/4}^{\pi/2} \frac{\sin(\frac{\pi}{4} - x)}{x^k}\, dx.$$

The substitutions $x = \frac{\pi}{4} - u$ in the first integral and $x = \frac{\pi}{4} + u$ in the second one lead to

$$\frac{\sqrt{2}}{2} \cdot G(\pi/2) = \int_0^{\pi/4} (\sin u) \left(\frac{1}{(\frac{\pi}{4} - u)^k} - \frac{1}{(\frac{\pi}{4} + u)^k} \right) du.$$

But for $u \in (0, \frac{\pi}{4})$, we have

$$(\frac{\pi}{4} + u)^k \geq (\frac{\pi}{4} - u)^k > 0$$

and $\sin u > 0$, hence

$$(\sin u) \left(\frac{1}{(\frac{\pi}{4} - u)^k} - \frac{1}{(\frac{\pi}{4} + u)^k} \right) \geq 0$$

and so $\frac{\sqrt{2}}{2} \cdot G(\pi/2) \geq 0$ and we are done.