OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2017: 43(5), p. 194–195.

OC331. Find all triples of nonnegative integers (x, y, z) and $x \leq y$ such that

$$x^2 + y^2 = 3 \cdot 2016^z + 77.$$

Originally 2016 Greece National Olympiad Problem 1.

We received 4 solutions of which 2 solutions were correct and complete and 2 were incorrect. We present the solution by Steven Chow.

We distinguish two cases.

First, assume $z = 0$. Then the equation reduces to $x^2 + y^2 = 80$. Since $0 \leq x \leq y$, the equation is satisfied by $x = 4$ and $y = 8$, only.

Second, assume that $z \geq 1$. For any integer a, a^2 is congruent to 0, 1, 2, or 4 modulo 7, and a^2 is congruent to 0 modulo 7 if and only if a is congruent to 0 modulo 7. Since for $z \geq 1$, $3 \cdot 2016^z + 77 \equiv 0 \pmod 7$, it follows that $x^2 + y^2 \equiv 0 \pmod 7$. However, the only two remainders among 0, 1, 2, 4 that sum to 0 (mod 7) are 0, 0; therefore $x \equiv 0 \pmod 7$ and $y \equiv 0 \pmod 7$. Let x_1 and y_1 be non-negative integers such that $x = 7x_1$ and $y = 7y_1$. In terms of x_1, y_1, and z, the equation becomes $7^2 \left(x_1^2 + y_1^2\right) = 3 \cdot 2016^z + 77$. The left side is divisible by 7^2. However, 77 is not divisible by 7^2, and the right side is divisible by 7^2 if only if $z = 1$. It follows that $z = 1$. Therefore $x_1^2 + y_1^2 = (3 \cdot 2016 + 77)/7^2 = 125$. Since $0 \leq x_1 \leq y_1$, (x_1, y_1) are either $(2, 11)$ or $(5, 10)$. This implies that (x, y) are either $(14, 77)$ or $(35, 70)$.

All non-negative integer solutions (x, y, z) of the given equation are the triplets $(4, 8, 0)$, $(14, 77, 1)$, and $(35, 70, 1)$.

OC332. Let $ABCD$ be a convex quadrilateral. Show that there exists a square $A'B'C'D'$ (where vertices may be ordered clockwise or counter-clockwise) such that $A \neq A', B \neq B', C \neq C', D \neq D'$ and AA', BB', CC', DD' are all concurrent.

Originally Problem 5 of Day 2 of the 2016 China National Olympiad.

We received only one solution that was incomplete.

OC333. Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

Originally 2016 USAMO Day 2 Problem 4.

We received 2 correct solutions. We present the solution by Mohammed Aassila.

We will establish that the only solutions are the functions $f(x) = 0$ and $f(x) = x^2$.

We evaluate the statement equation at specific points to obtain several properties of the function.

At $x = 0$ and $y = 0$: $2(f(0))^2 = (f(0))^2$ implies $f(0) = 0$.

At $x = 0$ and y arbitrary: $f(0)f(-3y)+f(y)f(-y) = (f(y))^2$ implies $f(y)f(-y) = (f(y))^2$. Also $f(y)f(-y) = (f(-y))^2$, from which we conclude $f(y) = f(-y)$, equivalently f is even.

At x arbitrary and $y = -x$: $(f(x) - x^2)f(4x) + (f(-x) - x^2)f(4x) = (f(0))^2$ implies that for arbitrary x

$$f(4x) = 0, \quad \text{or} \quad f(x) = x^2. \tag{1}$$

Let $t \neq 0$ such that $f(t) \neq 0$, then $f(t/4) = t^2/16$. At $x = t/4$ and $y = 3t/4$: $(f(t/4)+3t^2/16)f(-2t)+(f(3t/4)+3t^2/16)f(0) = (f(t))^2$ implies $t^2 f(2t)/4 = f(t)$. Consequently,

$$f(t) \neq 0 \quad \text{implies} \quad f(2t) \neq 0, f(4t) \neq 0, f(8t) \neq 0, \ldots \tag{2}$$

In addition, because $f(t) \neq 0$ implies $f(4t) \neq 0$ and because of (1), it follows that $f(t) = t^2$. Hence for arbitrary x

$$f(x) = 0, \quad \text{or} \quad f(x) = x^2. \tag{3}$$

Next, we shall prove that either $f(x) = 0$ for all x's or $f(x) = x^2$ for all x's. Assume the contrary: there exist $a \neq 0$ and $b \neq 0$ such that $f(a) = 0$ and $f(b) = b^2$. Evaluate the statement equation at $x = (a+b)/4$ and $y = (-a+3b)/4$:

$$\begin{aligned} &\left(f\left(\frac{a+b}{4}\right) + \frac{(a+b)(-a+3b)}{16}\right) f\left(a-2b\right) \\ &+ \left(f\left(\frac{-a+3b}{4}\right) + \frac{(a+b)(-a+3b)}{16}\right) f\left(a\right) = \left(f(b)\right)^2 \end{aligned}$$

or, equivalently,

$$\left(f\left(\frac{a+b}{4}\right) + \frac{(a+b)(-a+3b)}{16}\right) f\left(a-2b\right) = \left(f(b)\right)^2. \tag{4}$$

Relation (4) implies that if $f(b) = b^2$, then $f(a-2b) \neq 0$, $f(a-2b) = (a-2b)^2$, and then b and a are related via two polynomial equations:

$$\left((a+b)^2 + (a+b)(-a+3b)\right)\left(a-2b\right)^2 = 16b^2, \quad \text{or}$$

$$(a+b)(-a+3b)\left(a-2b\right)^2 = 16b^2. \tag{5}$$

Since a polynomial has finitely many roots, there exist finitely many b such that $f(b) \neq 0$. However, this contradicts (2) that states the existence of infinitely many b's such that $f(b) \neq 0$. Therefore our assumption that there exist $a \neq 0$ and $b \neq 0$ such that $f(a) = 0$ and $f(b) = b^2$ is incorrect.

In conclusion, there are only two solutions for the statement equation: $f(x) = 0$ for all x, or $f(x) = x^2$ for all x.

OC334. Let p be an odd prime number. For positive integers k satisfying $1 \leq k \leq p-1$, the number of divisors of $kp+1$ between k and p exclusive is a_k. Find the value of $a_1 + a_2 + \cdots + a_{p-1}$.

Originally 2016 Japan Mathematical Olympiad Finals Problem 1.

We received 2 solutions of which 1 was correct and 1 was incomplete. We present the solution by Steven Chow slightly modified by the editor.

For each $1 \leq k \leq p-1$, let A_k be the set consisting of the divisors of $kp+1$ between k and p exclusive. Therefore for each k, $a_k = |A_k|$ and

$$a_1 + \cdots + a_{p-1} = |A_1| + \cdots + |A_{p-1}|.$$

We establish two facts about the sets $A_1, \ldots, A_{p-1}$.

First, $A_1, \ldots, A_{p-1}$ have no common elements, equivalently are mutually exclusive. Assume the opposite that there exists integers k_1, k_2, and j such that

$$1 \leq k_1 \leq p-1, 1 \leq k_2 \leq p-1, k_1 < k_2 < j < p$$

and j belongs to the intersection of A_{k_1} and A_{k_2}. It follows that j is a common divisor of k_1p+1 and k_2p+1, and consequently j is a divisor of the difference

$$(k_2p+1) - (k_1p+1) = (k_2 - k_1)\, p.$$

Since p is prime and $j < p$ we can conclude that j is a divisor of $k_2 - k_1$. However this implies $j \leq k_2 - k_1 < k_2 < j$, which is a contradiction. Therefore the sets $A_1, \ldots, A_{p-1}$ are mutually exclusive.

Second, the union of the sets $A_1, \ldots, A_{p-1}$ is $\{2, 3, \ldots, p-1\}$. Let j be an integer such that $2 \leq j \leq p-1$. Since j and p are relatively prime integers, there exist two integers k and m such that $kp = mj - 1$ and $1 \leq k \leq j-1$. A short argument for this fact is provided in the editor's comments at the end of the solution. Rearranging the last equation into $kp+1 = mj$, we find that j is a divisor of $kp+1$, and hence $j \in A_k$. Since j was arbitrary, the union $A_1 \cup \cdots \cup A_{p-1}$ is $\{2, 3, \ldots, p-1\}$, a set with $p-2$ elements.

In conclusion $p - 2 = |A_1 \cup \cdots \cup A_{p-1}| = |A_1| + \cdots + |A_{p-1}| = a_1 + \cdots + a_{p-1}$.

Editor's comments. Regardless of the value of p, we have $a_{p-1} = 0$ as there are no integers between $p-1$ and p. Therefore the question could have been asked with $1 \leq k \leq p-2$ and led to the conclusion that $p - 2 = a_1 + \cdots + a_{p-2}$.

The proof used the fact that if j and p are relatively prime integers then there exist two integers k and m such that $kp = mj - 1$ and $1 \leq k \leq j-1$. This can be obtained by looking at the remainders of $p, 2p, \ldots, (j-1)p$ when divided by j. There are $j-1$ such remainders, all different and taking only $j-1$ values: $1, \ldots, j-1$. Therefore there exists k such that $1 \leq k \leq j-1$ and the remainder of kp when divided by j is $j-1$. Equivalently, there exists k and m such that $1 \leq k \leq j-1$ and $kp = mj + j - 1$, or $kp = (m+1)j - 1$.

OC335. Medians AM_A, BM_B and CM_C of a triangle ABC intersect at M. Let Ω_A be the circumcircle of the triangle that passes through the midpoint of AM and is tangent to BC at M_A. Define Ω_B and Ω_C analogously. Prove that Ω_A, Ω_B and Ω_C intersect at one point.

Originally 2016 All Russian Olympiad Grade 11 Day 2 Problem 8.

We received 2 solutions of which 1 was correct and 1 was incomplete. We present the solution by Steven Chow slightly modified by the editor.

We use barycentric coordinates. Let $(1,0,0) = A$, $(0,1,0) = B$, and $(0,0,1) = C$. Let $a = BC$, $b = CA$, and $c = AB$. Let x, y, and z represent the first, second and third coordinate of an arbitrary point, respectively. Let u, v, and w be the numbers such that the equation of Ω_A is

$$0 = -a^2yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z).$$

Since $M_A = (0, 1/2, 1/2)$ is on Ω_A, $v + w = a^2/2$.

Since Ω_A is tangent to the line BC described by $x = 0$, the equation

$$0 = -a^2yz + (vy + wz)(y + z)$$

has unique solution y and z. This is equivalent to the quadratic equation

$$0 = v(y/z)^2 - (a^2/2)(y/z) + w$$

in y/z having unique solution, or the equation having zero discriminant. The zero discriminant implies that $vw = \left(a^2/4\right)^2$. Therefore $v = w = a^2/4$.

Since the midpoint of segment AM has barycentric coordinates $(4/6, 1/6, 1/6)$ and lies on Ω_A,

$$0 = -a^2 - 4b^2 - 4c^2 + \left(4u + a^2/2\right)(6),$$

which leads to $u = -a^2/12 + b^2/6 + c^2/6$.

Therefore the equation of Ω_A is

$$0 = -a^2yz - b^2zx - c^2xy + \left(\left(-\frac{a^2}{12} + \frac{b^2}{6} + \frac{c^2}{6}\right)x + \frac{a^2}{4}y + \frac{a^2}{4}z\right)(x + y + z). \quad (1)$$

Similarly, the equations of Ω_B and Ω_C are the cyclic forms of (1).

Next, we claim that the point Ω with barycentric coordinates in proportion

$$\left(\left(-2a^2+b^2+c^2\right)^2 : \left(a^2-2b^2+c^2\right)^2 : \left(a^2+b^2-2c^2\right)^2\right)$$

satisfies the equation of Ω_A, and hence lies on Ω_A. Similarly, by cyclicity, this point is also on Ω_B and Ω_C, and so is the intersection point of the three circles. The intersection point was identified by computing the radical centre of the three circles, which is the point of intersection of two of the three radical axes. The equation of a radical axis is computed by subtracting the equations of two circles.

To see that the point Ω satisfies the equation (1) of Ω_A, we introduce some notations $m=-2a^2+b^2+c^2$, $n=a^2-2b^2+c^2$, and $p=a^2+b^2-2c^2$. Note that $m+n+p=0$. Consequently, after taking the square of

$$m^2+n^2+p^2=-2(mn+np+pm),$$

we have:

$$\begin{aligned}\left(m^2+n^2+p^2\right)^2 &= 4(mn+np+pm)^2\\ &= 4\left(m^2n^2+n^2p^2+p^2m^2+2mnp\,(m+n+p)\right)\\ &= 4\left(m^2n^2+n^2p^2+p^2m^2\right). \qquad (2)\end{aligned}$$

Also, since $m=-(n+p)$,

$$m^2+n^2+p^2=(n+p)^2+n^2+p^2=2(n^2+p^2+np). \qquad (3)$$

The coordinates of Ω are in proportion $\left(m^2:n^2:p^2\right)$. We evaluate the second part of the equation (1) of Ω_A at $x_\Omega=m^2$, $y_\Omega=n^2$, and $z_\Omega=p^2$. We use (2), (3), and $m+n+p=0$ in the following computations:

$$\begin{aligned}&\left(\left(-\frac{a^2}{12}+\frac{b^2}{6}+\frac{c^2}{6}\right)x_\Omega+\frac{a^2}{4}y_\Omega+\frac{a^2}{4}z_\Omega\right)(x_\Omega+y_\Omega+z_\Omega)\\ &=\left(-\frac{a^2}{3}+\frac{b^2}{6}+\frac{c^2}{6}\right)x_\Omega(x_\Omega+y_\Omega+z_\Omega)+\frac{a^2}{4}(x_\Omega+y_\Omega+z_\Omega)^2\\ &=\frac{m}{6}m^2\left(m^2+n^2+p^2\right)+\frac{a^2}{4}\left(m^2+n^2+p^2\right)^2\\ &=\frac{m}{3}m^2\left(n^2+p^2+np\right)+a^2(m^2n^2+n^2p^2+p^2m^2)\\ &=a^2n^2p^2+\left(a^2+\frac{m}{3}\right)m^2p^2+\left(a^2+\frac{m}{3}\right)n^2m^2+m^2\frac{mnp}{3}\\ &=a^2n^2p^2+\left(b^2+\frac{n}{3}\right)m^2p^2+\left(c^2+\frac{p}{3}\right)n^2m^2+m^2\frac{mnp}{3}\\ &=a^2n^2p^2+b^2p^2m^2+c^2m^2n^2+\frac{m^2np}{3}(m+n+p)\\ &=a^2n^2p^2+b^2p^2m^2+c^2m^2n^2=a^2y_\Omega z_\Omega+b^2z_\Omega x_\Omega+c^2x_\Omega y_\Omega.\end{aligned}$$

Therefore Ω_A, Ω_B, and Ω_C intersect in one point.