Linear Recurrence Sequences and Polynomial Division in Number Theory

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1 Introduction

The paper discusses some characteristics of linear recurrence sequences. It is inspired by the article "Polynomial Division in Number Theory" by James Rickards (***Crux***, 43(10), December 2017.).

Let us look at the following Diophantine equation which is symmetric in the unknowns x and y:

$$x^2 + pxy + y^2 = q, \tag{1}$$

where p and q are integers, and $p > 2$ (if $p \leq 2$, then there can only be a finite number of integer solutions). The integer solutions for $p > 2$ satisfy a linear recurrence equation

$$a_{n+2} = pa_{n+1} - a_n, \tag{2}$$

whose terms are integer coordinates on the hyperbola with equation (1).

Theorem 1 *Let $p > 2$ in the Diophantine equation. Then for any solution (x, y):*

(i) *if $q < 0$, x and y always have the same sign.*

(ii) *if $q > 0$, x and y, if they exist, can have the same or opposite signs.*

Theorem 2 *If the Diophantine equation (1) given $q \neq 0$ and $p > 2$ has solutions, then there is an infinite number of such solutions, which are pairs of neighbour terms of an integer sequence with linear recurrence equation (2).*

Proof. Let's assume that the equation (1) has a solution (x_1, y_1) in positive integers. By symmetry, we can assume without loss of generality that $x_1 \leq y_1$. Thus, considered as a quadratic in x, the equation $x^2 - pxy_1 + y_1^2 - q = 0$ has a root x_1. As $p > 2$, the discriminant $D = (p-4)\,y_1^2 + 4q > 0$, so there exists a second root x_2. By Vieta's formulas, we have $x_2 = py_1 - x_1$; as $p > 2$ and $x_1 \leq y_1$, we have $x_2 > y_1$.

Considering the equation $x_2^2 - px_2y + y^2 - q = 0$ as a quadratic in y, we find that a positive integer y_2 exists so that $y_2 > x_2$ and $x_2^2 - p_2x_2 + y_2^2 - q = 0$. Continuing this process, we obtain an increasing sequence of natural numbers. Every three neighbour terms respectively fulfil Vieta's formula. The process is known as "Vieta jumping", and yields an endless linear recurrence sequence of integers with a recurrence equation (2). Analogously, we can develop a jumping "downwards". Every two neighbour terms of the resulting sequence $\{a_n\}$ satisfy the Diophantine equation (1). $\square$

Figure 1 shows the two branches of the hyperbola $x^2 - 3xy + y^2 = 1$ (an example with $q > 0$). The integer points on its graph have coordinates that are pairs of consecutive terms of the sequence $\{\dots, -8, -3, -1, 0, 1, 3, 8, \dots\}$ satisfying $a_{n+2} = 3a_{n+1} - a_n$. Figure 2 shows an example with $q < 0$, the hyperbola $x^2 - 3xy + y^2 = -1$. The integer points have coordinates that are pairs of consecutive terms of the sequence $\{\dots, 5, 2, 1, 1, 2, 5, \dots\}$, which also satisfies $a_{n+2} = 3a_{n+1} - a_n$.

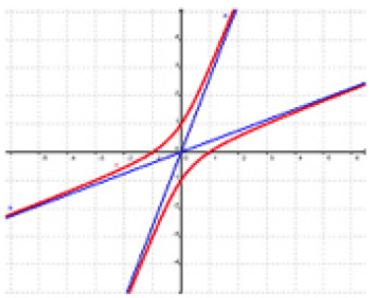


Figure 1

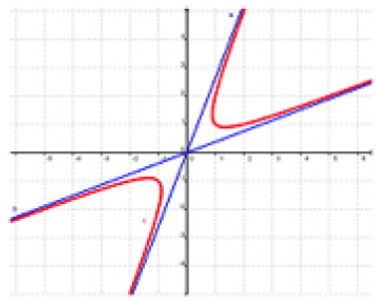


Figure 2

Theorem 3 *If the Diophantine equation* (1) *with* $p > 2$, $q > 0$ *has integer solutions, then* q *is a perfect square if and only if there is an integer point on the hyperbola, one of whose coordinates is equal to zero.*

Criterion 1 *If the Diophantine equation* (1) *with* $p > 2$, $q > 0$, *and* $q \le p$ *has a solution, then* q *is the perfect square of an integer.*

Below, we will apply these results to some contest problems.

Problem 1 (1988 International Mathematical Olympiad) *Let* a *and* b *be such positive integers that* $ab + 1$ *divides* $a^2 + b^2$. *Prove that the number* $\dfrac{a^2 + b^2}{ab + 1}$ *is a perfect square.*

(The problem was considered the hardest at the 29th IMO. It is said that five number theorists from Australia were not able to solve the problem in five hours! With modern techniques, this problem does not look that tough.)

Solution. We must solve the Diophantine equation $a^2 - pab + b^2 = q$ for $p = q = 1$. According to the formulated criterion, if this equation has a solution in integers, then p is a perfect square of an integer. All solutions are given by a linear recurrence sequence which includes the number zero.

Let $p = m^2$. Since we have already concluded that zero is part of the sequence, then by "Vieta Jumping" the pair $(0, m)$ yields a sequence of polynomials $0, m, m^2, m^3, m^5 - m, m^7 - 2m^3, \dots$, obtained through the recurrence relation $a_{n+1} = m^2 a_n - a_{n-1}$. This sequence of polynomials provides all solutions of the problem. The hyperbola is of the type illustrated in Figure 1, and both branches cross the coordinate axis in integer points. □

Problem 2 (1998 Canadian Mathematical Olympiad, National Round)

Let the sequence $\{a_n\}$ be defined by $a_0 = 0$, $a_1 = m$, $a_{n+1} = m^2 a_n - a_{n-1}$. Prove that all integer solutions (a, b) of the equation $\dfrac{a^2+b^2}{ab+1} = m$ given $a < b$ coincide with the pairs (a_n, a_{n+1}).

Problem 3 (Kvant magazine, M1225) *Prove that if for the natural numbers a and b the number $\dfrac{a^2+b^2}{ab-1}$ is also natural, then $\dfrac{a^2+b^2}{ab-1} = 5$.*

Solution. Denote $\frac{a^2+b^2}{ab-1} = p$. Then $a^2 - pab - b^2 = -p$. As $a \neq b$ and $b > 0$, we reach a version of (1) with $q = -p < 0$. In this case, the hyperbola does not cross the coordinate axis, and the respective sequences of solutions, if such, include only positive or negative terms. Let's look at the sequence with positive numbers; let y_0 be its least element. The equation $x^2 - pxy_0 + y^2 = -p$ has integer solutions x_1 and x_2, according to the assumption. Let $y_0 < x_1 \leq x_2$, which means

$$\frac{1}{2}\left(py_0 - \sqrt{(p^2-4)\,y_0^2 - 4p}\right) > y_0.$$

Thus $p > 2$, $(p-2)y_0^2 < p$. As p and y_0 are natural numbers, $y_0 = 1$. The discriminant becomes $p^2 - 4 - 4p$; as it is a perfect square, and differs by 8 from the perfect square $p^2 + 4p - 4$, we must have $p = 5$. This way we find $x_1 = 2$, $x_2 = 3$ and deduce that the Diophantine equation $x^2 - 5xy + y^2 + 5 = 0$ has an infinite number of solutions in natural numbers. These are adjacent pairs of terms from the sequence $\{\dots, 14, 3, 1, 2, 9, \dots\}$ with linear recurrence equation $a_{i+2} = 5a_{i+1} - a_i$. They are also the pairs of coordinates of the integer points on the hyperbola $x^2 - 5xy + y^2 + 5 = 0$. □

Problem 4 (2007 Spain Mathematical Olympiad, National Round) *Find all the possible positive integers which the expression $\dfrac{m^2+mn+n^2}{mn-1}$ can take where m and n are natural numbers and $mn \neq 1$.*

Problem 5 (2013 British Mathematical Olympiad) *Find all pairs of natural numbers x and y for which x divides y^2+1 and y divides x^2+1.*

Problem 6 (2012 Vietnam Mathematical Olympiad) *Let a and b be two odd natural numbers, where a is a divisor of b^2+2, and b is a divisor of a^2+2. Prove that a and b are terms of the sequence of natural numbers $\{v_n\}$ for which $v_1 = v_2 = 1$, $v_n = 4v_{n-1} - v_{n-2}$ if $n \geq 3$.*

Problem 7 (1999 Bulgaria Team Selection Test) *Prove that the number $n^4 + 1$ has a divisor of the $nm-1$ type (m and n are natural numbers) if and only if n is a term of the sequence $\{a_i\}_{i=1}^n$ for which $a_1 = 1$, $a_2 = 2$ or $a_2 = 3$, $a_{i+2} = 5a_{i+1} - a_i$.*

Comment. This problem is based on the fact that the defined sequence $\{a_i\}_{i=1}^n$ has the property $a_k^4 + 1 = (a_{k-1}a_k - 1)(a_k a_{k+1} - 1)$. This property is peculiar to the specific sequence of this problem; it is not shared by all sequences with the linear recurrence equation $a_{i+2} = 5a_{i+1} - a_i$.

We finish with a pair of harder problems (with solutions).

Problem 8 (2007 International Mathematical Olympiad) *Let a and b be natural numbers for which $4ab-1$ divides $\left(4a^2-1\right)^2$. Prove that $a=b$.*

Solution. In this problem the relation $(a-b)^2 = k(4ab-1)$ is valid where $k \geq 0$ is an integer. This follows from the equation

$$\left(4a^2-1\right)^2 = \left(4a^2-4ab+4ab-1\right)^2 = 16a^2(a-b)^2 + (4ab-1)(8a^2-4ab-1),$$

when considered that $4ab-1$ and $16a^2$ are mutually prime. When $k=0$ we have $a=b$. Let's assume the possibility that $k \geq 1$, i.e. $a \neq b$. If integer solutions (a,b) exist, then they are pairs of consecutive terms of the linear recurrence sequence obtained through "Vieta Jumping". Let b_0 be the least positive integer in this sequence. From Vieta's formulas we can deduce that positive integers a_1 and a_2 exist so that the pairs (b_0, a_1) and (b_0, a_2) are solutions to the equation in this case. Moreover, $a_1 > b_0$ and $a_2 > b_0$, which means that $(a_1-1)(a_2-1) \geq b_0^2$, and so we obtain

$$b_0^2 + k - 2(2k+1)b_0 = a_1a_2 - (a_1+a_2) = (a_1-1)(a_2-1) - 1 \geq b_0^2 - 1,$$

whence $b_0 \leq \dfrac{k+1}{4k+2} \leq 1$, a contradiction. $\square$

Problem 9 (La Gaceta de la RSME, Vol. 17 (2014), Problema 241) *Find all positive integers a and b for which the expression $\dfrac{a^4-a^2+1}{ab-1}$ is a positive integer.*

Solution. If $a=1$, then $b=2$. Let $a>1$, and let a and b fulfil the condition of the problem. We can write $a^4-a^2+1 = (ab-1)\,B$; then $a^4-a^2 = abB-(B+1)$. Consequently, a divides $B+1$, so $a^4-a^2+1 = (ab-1)\,(ka-1)$. We rewrite the numerator:

$$\begin{aligned} a^4-a^2+1 &= \left(b^4-b^2+1\right)a^4 - \left(a^2b^2-1\right)\left(a^2b^2-a^2+1\right) \\ &= \left(b^4-b^2+1\right)a^4 - (ab-1)A, \end{aligned}$$

where A is a positive integer. As a^2 and $ab-1$ are mutually prime, b^4-b^2+1 is divisible by $ab-1$. In the same way, we find that k^4-k^2+1 is divisible by $ka-1$. Obviously $a \neq b$ and $a \neq k$. If $a<b$, then $k<a$, because otherwise we would find that $a^4-a^2+1 = (ab-1)(ka-1) \geq \left(a^2+a-1\right)^2$, which is impossible. Therefore

$$ab(ka-1) = a^4-a^2+1+(ka-1) \quad \text{or} \quad b = \frac{a^2+k^2-1}{ka-1}a - k = pa-k.$$

Then p is also a positive integer. We obtain the equation

$$a^2 - pka + k^2 + p - 1 - 0,$$

which is symmetric in relation to a and k (and has no solutions for $p \leq 2$). Let (a_0, k_0) be the integer solution minimizing the value of k_0 (note that $k_0 \leq a_0$), and let a_1 be the other root of

$$a^2 - pk_0a + k_0^2 + p - 1 - 0.$$

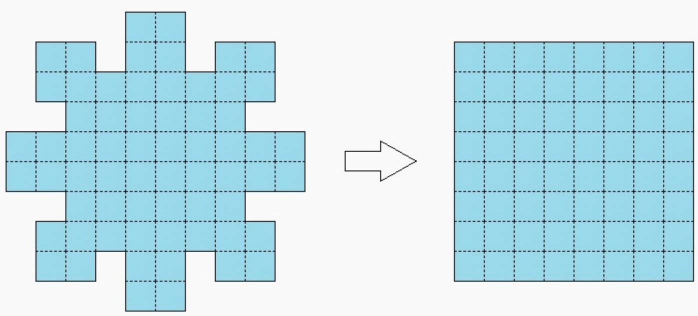
We can consider $a_1 \geq a_0 > k_0$. The root a_1 also is a positive integer due to the relation $a_1 = pk_0 - a_0$. Then

$$k_0 < a_0 = \frac{pk_0 - \sqrt{p^2k_0^2 - 4k_0^2 - 4p + 4}}{2}.$$

The only possibility is $k_0 = 1$, from where it follows that $p = 4$. This way we find $a_0 = 2$ and $a_1 = 2$. Because a, b, and k are symmetrical, we conclude that the solutions a, b of the problem are consecutive terms of the sequence $\{1, 2, 7, 26, \dots\}$ with linear recurrence equation $x_{n+2} = 4x_{n+1} - x_n$. □

Quadrature of the figure

Take a grid paper and cut out the figure shown below on the left. Can you cut it into 5 pieces and arrange them to form an 8×8 square?



Puzzle by Nikolai Avilov.