FOCUS ON...

No. 32

Michel Bataille

Harmonic Ranges and Pencils

Introduction

Elementary properties of harmonic conjugacy can lead to simple and elegant solutions to some geometry problems. Before considering examples, let us recall the basic definitions. Let A, B, C, D be four distinct points on a line. We say that C, D are harmonic conjugates w.r.t A, B when C, D divide AB in the same ratio, that is, if $\frac{\overline{CA}}{\overline{CB}} = -\frac{\overline{DA}}{\overline{DB}}$ (here and in what follows, the bar indicates signed distance and w.r.t. means "with respect to"). Clearly, the latter is equivalent to $\frac{\overline{AD}}{\overline{AC}} = -\frac{\overline{BD}}{\overline{BC}}$, meaning that A, B are harmonic conjugates w.r.t. C, D. If either condition is satisfied, we say that A, B, C, D is a harmonic division or a harmonic range. Let I be the midpoint of AB. Starting with $\overrightarrow{AD} = -k\overrightarrow{AC}$ and $\overrightarrow{BD} = k\overrightarrow{BC}$ for some real number k, easy manipulations give $\overrightarrow{ID} = k\overrightarrow{IA}$ and $\overrightarrow{IA} = k\overrightarrow{IC}$ and conversely. Thus, the condition $IA^2 = \overline{IC}.\overline{ID}$ can also be used to prove the harmonicity of the range of collinear points A, B, C, D.

Harmonic pencil

Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four distinct lines which are either parallel or concurrent, and let transversals m, m' meet them in A, B, C, D and in A', B', C', D', respectively. If $\ell_1, \ell_2, \ell_3, \ell_3$ are parallel, then $\frac{\overline{C'A'}}{\overline{C'B'}} = \frac{\overline{CA}}{\overline{CB}}$ and $\frac{\overline{D'A'}}{\overline{D'B'}} = \frac{\overline{DA}}{\overline{DB}}$ so that A', B', C', D' is a harmonic range as soon as A, B, C, D is one (Figure 1).

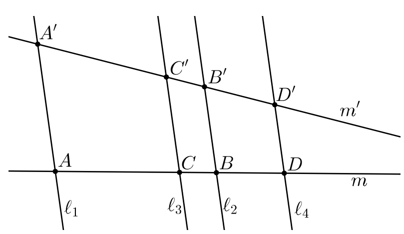

Figure 1

This conservation of harmonicity remains true when $\ell_1, \ell_2, \ell_3, \ell_4$ are concurrent lines. To prove this, we shall use the following lemma (for a proof, we refer the reader to **[1]** p. 169).

> Let A, B, C, D be four distinct points on a line and S a point not on this line. Let the parallel to SC through A intersect SD at M and SB at E. Then A, B, C, D is a harmonic range if and only if M is the midpoint of AE (Figure 2).

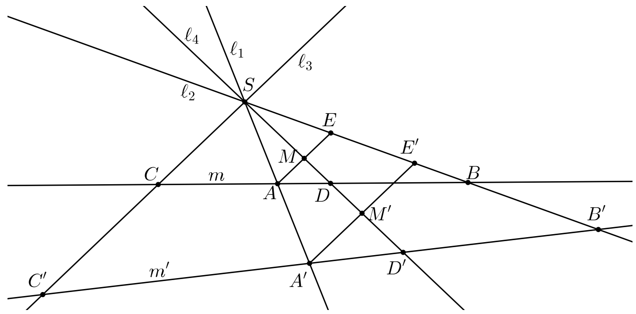

Figure 2

Now, let $\ell_1, \ell_2, \ell_3, \ell_4$ be concurrent at S and let m intersect them along a harmonic range A, B, C, D. If m' intersects them along A', B', C', D', we draw the parallels to SC through A and through A', which intersect SD and SB, respectively at M and E and at M' and E' (Figure 2). Since A, B, C, D is a harmonic range, M is the midpoint of AE; since AE is parallel to $A'E'$, M' is the midpoint of $A'E'$ and so A', B', C', D' is a harmonic range as well.

This justifies the following definition: $\ell_1, \ell_2, \ell_3, \ell_4$ is called a harmonic pencil when a transversal m intersects $\ell_1, \ell_2, \ell_3, \ell_4$ along a harmonic range. From the lemma above, an example is given by the lines ℓ, AM, AB, AC if M is the midpoint of the side BC of ΔABC and ℓ is the parallel to BC through A.

We are now ready to examine a few situations involving harmonic ranges or pencils and illustrate them with problems.

An angle and its bisectors

If ABC is a triangle (with $AB \neq AC$) and the internal and external bisectors of $\angle BAC$ meet BC at D and D', respectively, we know that D, D' divide BC in the ratio $\frac{AB}{AC}$. Thus, B, C, D, D' is a harmonic range and AB, AC, AD, AD' is a harmonic pencil. Note that AD, AD' are perpendicular. Interestingly, a kind of converse holds (easily proved or see [**1**] p. 170):

> Let $\ell_1, \ell_2, \ell_3, \ell_4$ be a harmonic pencil of concurrent lines at S. If ℓ_3, ℓ_4 are perpendicular, then they are the axes of symmetry of ℓ_1, ℓ_2.

To illustrate these results, we consider problem **3036** [2005 : 175 ; 2006 : 244], slightly modified:

> Let A, B, C be three distinct collinear fixed points. Let M be an arbitrary point not on the line ABC. The internal angle bisector of $\angle MAB$ intersects the line MB at a point X. The perpendicular at A to the line AX intersects the line MC at a point Y.
>
> (a) Prove that the line XY passes through a fixed point D.
>
> (b) Let Z be the projection of the point A onto the line XY. Prove that ZA is a bisector of $\angle BZC$.

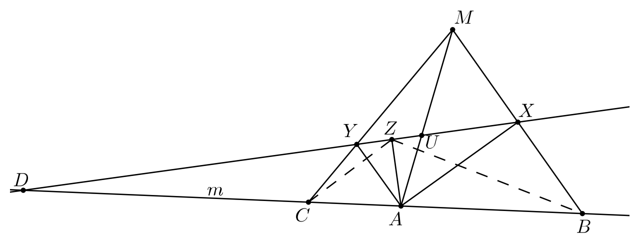

Figure 3

(a) Let XY intersect MA at U and the line m through A, B, C at D (Figure 3). Since AY and AX are the bisectors of $\angle MAB$, the pencil AY, AX, AM, AB is harmonic. Therefore, the range Y, X, U, D is harmonic and in consequence the pencil MY,MB,MU,MD is harmonic. Finally, considering the latter and the transversal m, we see that C, B, A, D is harmonic and conclude that XY always passes through the harmonic conjugate of A w.r.t. B, C.

(b) Since ZA and ZD are perpendicular and C, B, A, D is a harmonic range, ZA, ZD are the bisectors of $\angle BZC$.

About polars with respect to a circle

Consider a circle Γ with centre O and radius r and let M be a point distinct from O. The locus Π_M of points P such that the dot product $\overrightarrow{OM} \cdot \overrightarrow{OP}$ equals r^2 is the polar of M w.r.t. Γ. If M lies on Γ, M itself is a point of Π_M; otherwise, denoting by A and B the points of intersection of Γ and the line OM, we see that the harmonic conjugate M' of M w.r.t. A, B is a point of Π_M (since O is the midpoint of AB and $\overline{OM}.\overline{OM'} = OA^2$). Moreover, P is on Π_M if and only if $\overrightarrow{OM} \cdot \overrightarrow{OP} = \overrightarrow{OM} \cdot \overrightarrow{OM'}$, which is equivalent to $\overrightarrow{OM} \cdot \overrightarrow{M'P} = 0$, and therefore Π_M is the perpendicular to OM through M' (Figure 4). In the same way, we obtain that if M is on Γ, then Π_M is the tangent to Γ at M.

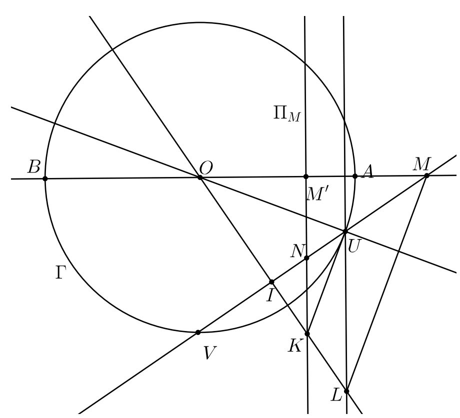

Figure 4

In passing, note that M' is the inverse of M in Γ and that M is on Π_N as soon as N is on Π_M (polar reciprocity).

The result that M, M', A, B is a harmonic range can be generalized as follows:

> If a line through M intersects Π_M at N and the circle Γ at U and V, then M, N, U, V is a harmonic range.

Let the perpendicular bisector of UV intersect UV at I, Π_M at K and the perpendicular to OM through U at L (Figure 4). Clearly, U is the orthocenter of ΔOML, hence OU is perpendicular to ML. But M is on Π_K (since K is on Π_M) and so UV is the polar of K. It follows that K is on the polar of U, that is, the tangent to Γ at U. Being both perpendicular to OU, ML and KU are parallel, and so are KN and LU. As a result, if $\overrightarrow{IU} = k\overrightarrow{IN}$, then $\overrightarrow{IL} = k\overrightarrow{IK}$ and so $\overrightarrow{IM} = k\overrightarrow{IU}$. The result follows.

As an application, we propose here a problem of the 50th Olympiad of Moldova [2009 : 377]:

> The quadrilateral $ABCD$ is inscribed in a circle. The tangents to the circle at A and C intersect at a point P not on BD and such that $PA^2 = PB.PD$. Prove that BD passes through the midpoint of AC.

Let Γ be the circumcircle of $ABCD$ and let O be its centre. The line PD intersects Γ again at B' with $B' \neq B$ (since P is not on BD). Since $PB' \cdot PD$ is the power of P w.r.t. Γ, we have $PB' \cdot PD = PA^2 = PB \cdot PD$, so that $PB' = PB$ and the line OP is the perpendicular bisector of BB' (Figure 5).

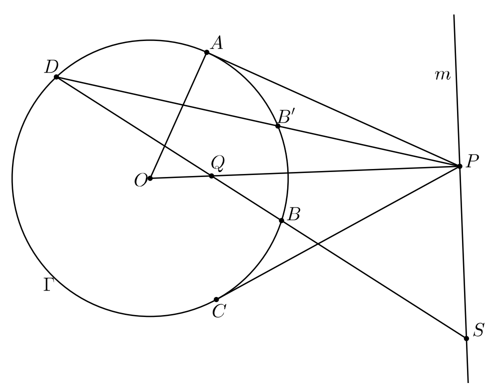

Figure 5

It follows that OP is a bisector of the angle $\angle BPD$ and so is the line m perpendicular to OP at P. As a result, PO, m, PD, PB is a harmonic pencil and the line BD intersects PO and m at Q and S such that Q, S, B, D is a harmonic range. From the property above, we then deduce that the polar of Q w.r.t. Γ passes through S, hence is m (since m is perpendicular to OQ). By polar reciprocity, Q is on the polar of P, which is AC, and the conclusion immediately follows.

Constructions with the straightedge alone

Harmonic ranges or pencils can be constructed with the straightedge alone. This interesting feature rests upon the following property:

> Let ℓ_1, ℓ_2 be two lines intersecting at S and A a point not on these lines. Through A we draw two transversals intersecting ℓ_1, ℓ_2 at M_1, M_2 and N_1, N_2. If the lines M_1N_2 and M_2N_1 intersect at U, then SA, SU, ℓ_1, ℓ_2 is a harmonic pencil.

The proof is easy: If B is the harmonic conjugate of A w.r.t. M_1, M_2, the line through A, N_1, N_2 is a transversal of the harmonic pencil UA, UB, UM_1, UM_2, hence intersects UB at C such that A, C, N_2, N_1 is a harmonic range. The line SB, which also passes through C, must coincide with SU.

Of course, if ℓ_1, ℓ_2 are parallel, a similar conclusion holds provided that SA and SU are replaced by the parallels to ℓ_1, ℓ_2 through A and U, respectively.

To see this at work, a good example is Problem **2965** [2004 : 367, 370; 2005 : 405]:

> Let $ABCD$ be a parallelogram. Using only an unmarked straightedge, find a point M on AB such that $AM = \frac{1}{5}AB$.

Here are the steps of the construction. First, we obtain the reflection B_1 of B in A by drawing ℓ such that DC, DA, DB, ℓ is a harmonic pencil (Figure 6a). The line ℓ intersects the line AB at B_1. Second, we construct B_1' such that A, B, B_1, B_1' is a harmonic range (Figure 6b). Finally, we repeat the first two steps with B_1' instead of B. This yields the desired point M as the harmonic conjugate w.r.t. A, B of the reflection B_2 of B_1' in A.

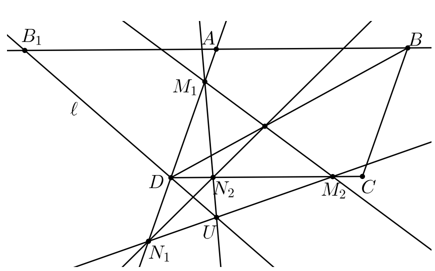

Figure 6a

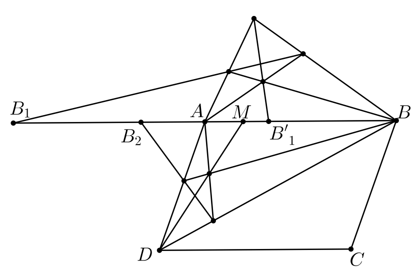

Figure 6b

Indeed, we have $\frac{\overline{B_1'A}}{\overline{B_1'B}} = -\frac{\overline{B_1A}}{\overline{B_1B}} = -\frac{1}{2}$ and so

$$\frac{\overline{AM}}{\overline{MB}} = \frac{\overline{B_2A}}{\overline{B_2B}} = -\frac{\overline{B_1'A}}{\overline{AB_1'} + \overline{AB}} = \frac{-\overline{B_1'A}/\overline{B_1'B}}{-\overline{B_1'A}/\overline{B_1'B} + \overline{AB}/\overline{B_1'B}} = \frac{1/2}{1/2 + 3/2} = \frac{1}{4}$$

The relation $AM = \frac{1}{5}AB$ follows.

Exercises

1. Through a point P exterior to a given circle pass a secant and a tangent to the circle. The secant intersects the circle at A and B, and the tangent touches the circle at C on the same side of the diameter through P as A and B. The projection of C onto the diameter is Q. Prove that QC bisects $\angle AQB$. (Set at the competition Baltic Way in 2004.)

2. The standard construction for bisecting a line segment involves the use of two arcs and one straight line. Show that it can, in fact, be done with straight lines and just one arc. (Problem 88.I of the Mathematical Gazette, proposed in November 2004.)

Reference

[1] N. Altshiller-Court, *College Geometry*, Dover, 2007.