Crux Mathematicorum

VOLUME 44, NO. 7 September / Septembre 2018

Editorial Board

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THE CONTEST CORNER

No. 67

John McLoughlin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **February 1, 2019**.

The editor thanks Valérie Lapointe, Carignan, QC, for translations of the problems.

CC331. Consider triangle ABC with $\angle B = \angle C = 70^\circ$. On the sides AB and AC, we take the points F and E, respectively, so that $\angle ABE = 15^\circ$ and $\angle ACF = 30^\circ$. Find $\angle AEF$.

CC332. Find the largest integer k such that 135^k divides 2016!. Note that $n! = 1 \cdot 2 \cdot 3 \cdots n$.

CC333. Let $\theta = \arctan 2 + \arctan 3$. Find $\dfrac{1}{\sin^2 \theta}$ and simplify fully.

CC334. Find the sum of all positive integers x for which $x + 56$ and $x + 113$ are perfect squares.

CC335. In the triangle ABC, BD is the median to the side AC, DG is parallel to the base BC (G is the point of intersection of the parallel with AB). In the triangle ABD, AE is the median to the side BD and F is the intersection point of DG and AE. Find $\dfrac{BC}{FG}$.

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CC331. Considérez le triangle ABC tel que $\angle B = \angle C = 70^\circ$. Sur les côtés AB et AC, on prend les points F and E tels que $\angle ABE = 15^\circ$ and $\angle ACF = 30^\circ$. Trouvez $\angle AEF$.

CC332. Trouvez le plus grand entier k tel que 135^k divise 2016!, où $n! = 1 \cdot 2 \cdot 3 \cdots n$.

CC333. Soit $\theta = \arctan 2 + \arctan 3$. Trouvez $\dfrac{1}{\sin^2 \theta}$ et simplifiez le plus possible.

CC334. Trouvez la somme de tous les entiers positifs x tels que $x + 56$ et $x + 113$ sont des carrés parfaits.

CC335. Dans le triangle ABC, BD est la médiane du côté AC et DG est parallèle à la base BC (G est le point d'intersection de la droite parallèle avec le côté AB). Dans le triangle ABD, AE est la médiane du côté BD et F est le point d'intersection des segments DG et AE. Trouvez le rapport $\dfrac{BC}{FG}$.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2017: 43(7), p. 281–281.

CC281. In the Original Six era of the NHL, one particular season, each team played 20 games (each team played the other 5 teams 4 times each). Each game ended as a win, a loss or a tie (there were no 'overtime losses'). At the end of this certain season, the standings were as below. What were all the possible outcomes for Montreal's number of wins X, losses Y and ties Z?

Team	Wins	Losses	Ties
Toronto	2	12	6
Boston	6	10	4
Detroit	7	12	1
New York	7	9	4
Chicago	11	7	2
Montreal	x	y	z

Originally Question 6 from the 2015 W.J. Blundon Mathematics Contest.

We received 8 solutions, all of which were correct and complete. We present the solution by Fernando Ballesta Yagüe.

The known results are:

- $2 + 6 + 7 + 7 + 11 = 33$ wins.
- $12 + 10 + 12 + 9 + 7 = 50$ losses.
- $6 + 4 + 1 + 4 + 2 = 17$ ties.

As we eventually need to have the same total number of wins and losses (if a team loses a match, another team has had to win that match), we have that $x + 33$, which is the total number of wins, has to be equal to $y + 50$, which is the total number of losses. Since the total number of matches played by Montreal's team is 20, we have:

$$x + y + z = 20,$$
$$x + 33 = y + 50.$$

From that we can deduce that

$$x = y + 17 \to (y + 17) + y + z = 20 \to z = 3 - 2y.$$

Since the total number of ties, which is $z + 17$, has to be even (if a match has ended in a tie, it has ended in a tie for both of the teams), we have that $z > 0$ and z is odd. The only two possibilities for $z > 0$ are $y = 0$ and $y = 1$. If $y = 0$, then $z = 3$ and $x = 17$; if $y = 10$, then $z = 1$ and $x = 18$. So the possible outcomes for Montreal's number of wins, losses and ties are:

$$x = 17,\ y = 0,\ z = 3$$
$$x = 18,\ y = 1,\ z = 1$$

CC282. Calculate the value of

$$\left(3^{4/3} - 3^{1/3}\right)^3 + \left(3^{5/3} - 3^{2/3}\right)^3 + \left(3^{6/3} - 3^{3/3}\right)^3 + \cdots + \left(3^{2006/3} - 3^{2003/3}\right)^3.$$

Originally Question 7 from the 2015 W.J. Blundon Mathematics Contest.

We received 20 submissions of which 17 were correct and complete. We present two solutions.

Solution 1, by Ivko Dimitrić.

Note that the difference of the two exponents of 3 of the pair of terms within each set of parentheses is 1, which means that the first term in each pair is 3 times the second one, i.e. those expressions are of the form

$$3^{(k+3)/3} - 3^{k/3} = 3 \cdot 3^{k/3} - 3^{k/3} = 2 \cdot 3^{k/3} = 2 \cdot \sqrt[3]{3^k},$$

for $k = 1, 2, \ldots, 2003$. Then the sum equals

$$\begin{aligned}
&(2 \cdot \sqrt[3]{3})^3 + (2 \cdot \sqrt[3]{3^2})^3 + (2 \cdot \sqrt[3]{3^3})^3 + \cdots + (2 \cdot \sqrt[3]{3^{2003}})^3 \\
&= 2^3 \cdot 3 + 2^3 \cdot 3^2 + 2^3 \cdot 3^3 + \cdots + 2^3 \cdot 3^{2003} \\
&= 8 \cdot (3 + 3^2 + 3^3 + \cdots + 3^{2003}) \\
&= 8 \cdot \frac{3\,(3^{2003} - 1)}{2} \\
&= 12\,(3^{2003} - 1).
\end{aligned}$$

Solution 2, by Miguel Amengual Covas.

The answer is $12\left(3^{2003}-1\right)$.

We give a generalization which, in the case $n = 2003$, yields the solution to the proposed problem.

$$\begin{aligned}
&\left(3^{\frac{4}{3}}-3^{\frac{1}{3}}\right)^3+\left(3^{\frac{5}{3}}-3^{\frac{2}{3}}\right)^3+\left(3^{\frac{6}{3}}-3^{\frac{3}{3}}\right)^3+\cdots+\left(3^{\frac{n+3}{3}}-3^{\frac{n}{3}}\right)^3 \qquad (1)\\
&=\left[3^{\frac{1}{3}}(3-1)\right]^3+\left[3^{\frac{2}{3}}(3-1)\right]^3+\left[3^{\frac{3}{3}}(3-1)\right]^3+\cdots+\left[3^{\frac{n}{3}}(3-1)\right]^3\\
&=3\cdot 2^3+3^2\cdot 2^3+3^3\cdot 2^3\cdots+3^n\cdot 2^3\\
&=2^3\left(3+3^2+3^3+\cdots+3^n\right).
\end{aligned}$$

The sum $3+3^2+3^3+\cdots+3^n$ is a geometric progression with value

$$\frac{3^n\cdot 3-3}{3-1}=\frac{3\left(3^n-1\right)}{2}.$$

Substituting this value in (1), we get

$$12\left(3^{2003}-1\right).$$

CC283. Two bags, Bag A and Bag B, each contain 9 balls. The 9 balls in each bag are numbered from 1 to 9. Suppose one ball is removed randomly from Bag A and another ball from Bag B. If X is the sum of the numbers on the balls left in Bag A and Y is the sum of the numbers of the balls remaining in Bag B, what is the probability that X and Y differ by a multiple of 4?

Originally Question 10 from the 2015 W.J. Blundon Mathematics Contest.

We received eight submissions to this problem, all of which were correct. We present the solution by Steven Chow, Miguel Amengual Covas, and Ballesta Yagüe Fernando (all done independently), modified by the editor.

Let x, y be the number of the ball removed from the Bag A and B, respectively. We have that

$$|X-Y|=|\,(45-x)-(45-y)\,|=|-x+y|.$$

X and Y differ by a multiple of 4 if and only if x and y are congruent modulo 4. From 1 to 9, the number of integers congruent to 0, 1, 2, 3 modulo 4 are 2, 3, 2, 2, respectively, so the probability that the numbers on the balls removed are congruent modulo 4 is

$$\left(\frac{2}{9}\right)^2+\left(\frac{3}{9}\right)^2+\left(\frac{2}{9}\right)^2+\left(\frac{2}{9}\right)^2=\frac{7}{27}.$$

Therefore the probability that X and Y differ by a multiple of 4 is $\frac{7}{27}$.

CC284. Define the function $f(x)$ to be the largest integer less than or equal to x for any real x. For example, $f(1) = 1, f(3/2) = 1, f(7/2) = 3, f(7/3) = 2$. Let

$$g(x) = f(x) + f(x/2) + f(x/3) + \cdots + f(x/(x-1)) + f(x/x).$$

a) Calculate $g(4) - g(3)$ and $g(7) - g(6)$.

b) What is $g(116) - g(115)$?

Originally Question 10 from the 2016 W.J. Blundon Mathematics Contest.

We received 8 correct solutions. We present the solution by Titu Zvonaru.

For positive integers$n$, we denote by $d(n)$ the number of divisors of n. Let $k = 1, 2, ..., n$. Dividing n by k, we have $n = pk + r$, with $0 \leq r < k$.

If $0 < r$, then $n - 1 = pk + (r - 1)$. Hence, $f\left(\frac{n}{k}\right) = f\left(\frac{n-1}{k}\right)$.

If $r = 0$, then $n-1 = (p-1)k+(k-1)$. Hence, $f\left(\frac{n}{k}\right) - f\left(\frac{n-1}{k}\right) = p-(p-1) = 1$.

We deduce that

$$g(n) - g(n-1) = d(n),$$

Since $4 = 2^2$, $7 = 7$, and $116 = 2^2 \cdot 29$, we have

$$\begin{aligned} g(4) - g(3) &= d(4) = 3 \\ g(7) - g(6) &= d(7) = 2 \\ g(116) - g(115) &= d(116) = 6. \end{aligned}$$

CC285. Find all values of k so that $x^2 + y^2 = k^2$ will intersect the circle with equation $(x-5)^2 + (y+12)^2 = 49$ at exactly one point.

Originally Question 6 from the 2016 W.J. Blundon Mathematics Contest.

We received 15 solutions, of which 10 were correct and complete and 5 were incomplete. One of the correct solutions was in Spanish. We present here the solution by Dan Daniel.

The two circles intersect in one point (internally or externally) if either $O_1O_2 = r_1 + r_2$ or $O_1O_2 = \|r_1 - r_2\|$, where we define O_1 as (0,0) and O_2 as (5,12). We then have $O_1O_2 = \sqrt{(5-0)^2 + (-12-0)^2} = 13$. So $r_1 = \|k\|$, and $r_2 = 7$.

The first case gives

$$\|k\| + 7 = 13 \implies \|k\| = 6 \implies k = \pm 6,$$

while the second gives

$$\|\|k\| - 7\| = 13 \implies \|k\| - 7 = \pm 13 \implies \|k\| = 20 \implies k = \pm 20.$$

Therefore $k \in \{-20, -6, 6, 20\}$.

THE OLYMPIAD CORNER

No. 365

Anamaria Savu

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **February 1, 2019**.

The editor thanks Valérie Lapointe, Carignan, QC, for translations of the problems.

OC391. Let $x_1, x_2, x_3, \ldots$ be a sequence of positive integers such that for every pair of positive integers (m, n) we have $x_{mn} \neq x_{m(n+1)}$. Prove that there exists a positive integer i such that $x_i \geq 2017$.

OC392. In a convex hexagon $ABCDEF$ all sides are equal and also $AD = BE = CF$. Prove that a circle can be inscribed into this hexagon.

OC393. The point O is the center of the circumcircle Ω of the acute triangle ABC. The circumcircle ω of the triangle AOC intersects the sides AB and BC again at the points E and F. Moreover, the line EF divides the area of the triangle ABC in half. Find $\angle B$.

OC394. In Chicago, there are 36 criminal gangs, some of which are at war with each other. Each gangster belongs to several gangs and every pair of gangsters belongs to a different set of gangs. It is known that no gangster is a member of two gangs that are at war with each other. Furthermore, each gang that some gangster does not belong to is at war with some gang he does belong to. What is the largest possible number of gangsters in Chicago?

OC395. Let $A_1, A_2, \ldots, A_k \in \mathcal{M}_n(\mathbb{R})$ be symmetric matrices. Prove that the following statements are equivalent:

(a) $\det(A_1^2 + A_2^2 + \cdots + A_k^2) = 0$;

(b) for all matrices $B_1, B_2, \ldots, B_k \in \mathcal{M}_n(\mathbb{R})$ it holds

$$\det(A_1B_1 + A_2B_2 + \cdots + A_kB_k) = 0.$$

OC391. Soit $x_1, x_2, x_3, \ldots$ une suite d'entiers positifs telle que pour chaque paire d'entiers positifs (m, n), on a $x_{mn} \neq x_{m(n+1)}$. Prouvez qu'il existe un entier positif i tel que $x_i \geq 2017$.

OC392. Soit un hexagone convexe $ABCDEF$ dont tous les côtés sont égaux et dont $AD = BE = CF$. Prouvez qu'un cercle peut être inscrit dans cet hexagone.

OC393. Le point O est le centre du cercle circonscrit Ω du triangle acutangle ABC. Le cercle circonscrit ω du triangle AOC intercepte les côtés AB et BC aux points E et F. De plus, le segment EF divise l'aire du triangle ABC en deux. Trouvez $\angle B$.

OC394. À Chicago, il y a 36 bandes criminelles, dont certaines sont en guerre une contre l'autre. Chaque bandit appartient à diverses bandes et chaque paire de bandits appartient à des groupes de bandes différents. Un bandit ne peut pas appartenir à deux bandes qui sont en guerre. De plus, chaque bande à laquelle un bandit n'appartient pas est en guerre avec certaines bandes auxquelles ce bandit appartient. Quel est le plus grand nombre possible de bandits à Chicago?

OC395. Soit $A_1, A_2, \ldots, A_k \in \mathcal{M}_n(\mathbb{R})$des matrices symétriques. Prouvez que les énoncés suivants sont équivalents :

(a) $\det(A_1^2 + A_2^2 + \cdots + A_k^2) = 0$;

(b) pour toutes matrices $B_1, B_2, \ldots, B_k \in \mathcal{M}_n(\mathbb{R})$ on a

$$\det(A_1B_1 + A_2B_2 + \cdots + A_kB_k) = 0.$$

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2017: 43(5), p. 194–195.

OC331. Find all triples of nonnegative integers (x, y, z) and $x \leq y$ such that

$$x^2 + y^2 = 3 \cdot 2016^z + 77.$$

Originally 2016 Greece National Olympiad Problem 1.

We received 4 solutions of which 2 solutions were correct and complete and 2 were incorrect. We present the solution by Steven Chow.

We distinguish two cases.

First, assume $z = 0$. Then the equation reduces to $x^2 + y^2 = 80$. Since $0 \leq x \leq y$, the equation is satisfied by $x = 4$ and $y = 8$, only.

Second, assume that $z \geq 1$. For any integer a, a^2 is congruent to 0, 1, 2, or 4 modulo 7, and a^2 is congruent to 0 modulo 7 if and only if a is congruent to 0 modulo 7. Since for $z \geq 1$, $3 \cdot 2016^z + 77 \equiv 0 \pmod 7$, it follows that $x^2 + y^2 \equiv 0 \pmod 7$. However, the only two remainders among 0, 1, 2, 4 that sum to 0 (mod 7) are 0, 0; therefore $x \equiv 0 \pmod 7$ and $y \equiv 0 \pmod 7$. Let x_1 and y_1 be non-negative integers such that $x = 7x_1$ and $y = 7y_1$. In terms of x_1, y_1, and z, the equation becomes $7^2 \left(x_1^2 + y_1^2\right) = 3 \cdot 2016^z + 77$. The left side is divisible by 7^2. However, 77 is not divisible by 7^2, and the right side is divisible by 7^2 if only if $z = 1$. It follows that $z = 1$. Therefore $x_1^2 + y_1^2 = (3 \cdot 2016 + 77)/7^2 = 125$. Since $0 \leq x_1 \leq y_1$, (x_1, y_1) are either $(2, 11)$ or $(5, 10)$. This implies that (x, y) are either $(14, 77)$ or $(35, 70)$.

All non-negative integer solutions (x, y, z) of the given equation are the triplets $(4, 8, 0)$, $(14, 77, 1)$, and $(35, 70, 1)$.

OC332. Let $ABCD$ be a convex quadrilateral. Show that there exists a square $A'B'C'D'$ (where vertices may be ordered clockwise or counter-clockwise) such that $A \neq A', B \neq B', C \neq C', D \neq D'$ and AA', BB', CC', DD' are all concurrent.

Originally Problem 5 of Day 2 of the 2016 China National Olympiad.

We received only one solution that was incomplete.

OC333. Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

Originally 2016 USAMO Day 2 Problem 4.

We received 2 correct solutions. We present the solution by Mohammed Aassila.

We will establish that the only solutions are the functions $f(x) = 0$ and $f(x) = x^2$.

We evaluate the statement equation at specific points to obtain several properties of the function.

At $x = 0$ and $y = 0$: $2(f(0))^2 = (f(0))^2$ implies $f(0) = 0$.

At $x = 0$ and y arbitrary: $f(0)f(-3y)+f(y)f(-y) = (f(y))^2$ implies $f(y)f(-y) = (f(y))^2$. Also $f(y)f(-y) = (f(-y))^2$, from which we conclude $f(y) = f(-y)$, equivalently f is even.

At x arbitrary and $y = -x$: $(f(x) - x^2)f(4x) + (f(-x) - x^2)f(4x) = (f(0))^2$ implies that for arbitrary x

$$f(4x) = 0, \quad \text{or} \quad f(x) = x^2. \tag{1}$$

Let $t \neq 0$ such that $f(t) \neq 0$, then $f(t/4) = t^2/16$. At $x = t/4$ and $y = 3t/4$: $(f(t/4)+3t^2/16)f(-2t)+(f(3t/4)+3t^2/16)f(0) = (f(t))^2$ implies $t^2 f(2t)/4 = f(t)$. Consequently,

$$f(t) \neq 0 \quad \text{implies} \quad f(2t) \neq 0, f(4t) \neq 0, f(8t) \neq 0, \ldots \tag{2}$$

In addition, because $f(t) \neq 0$ implies $f(4t) \neq 0$ and because of (1), it follows that $f(t) = t^2$. Hence for arbitrary x

$$f(x) = 0, \quad \text{or} \quad f(x) = x^2. \tag{3}$$

Next, we shall prove that either $f(x) = 0$ for all x's or $f(x) = x^2$ for all x's. Assume the contrary: there exist $a \neq 0$ and $b \neq 0$ such that $f(a) = 0$ and $f(b) = b^2$. Evaluate the statement equation at $x = (a+b)/4$ and $y = (-a+3b)/4$:

$$\begin{aligned} &\left(f\left(\frac{a+b}{4}\right) + \frac{(a+b)(-a+3b)}{16}\right) f\left(a-2b\right) \\ &+ \left(f\left(\frac{-a+3b}{4}\right) + \frac{(a+b)(-a+3b)}{16}\right) f\left(a\right) = \left(f(b)\right)^2 \end{aligned}$$

or, equivalently,

$$\left(f\left(\frac{a+b}{4}\right) + \frac{(a+b)(-a+3b)}{16}\right) f\left(a-2b\right) = \left(f(b)\right)^2. \tag{4}$$

Relation (4) implies that if $f(b) = b^2$, then $f(a-2b) \neq 0$, $f(a-2b) = (a-2b)^2$, and then b and a are related via two polynomial equations:

$$\left((a+b)^2 + (a+b)(-a+3b)\right)\left(a-2b\right)^2 = 16b^2, \quad \text{or}$$

$$(a+b)(-a+3b)\left(a-2b\right)^2 = 16b^2. \tag{5}$$

Since a polynomial has finitely many roots, there exist finitely many b such that $f(b) \neq 0$. However, this contradicts (2) that states the existence of infinitely many b's such that $f(b) \neq 0$. Therefore our assumption that there exist $a \neq 0$ and $b \neq 0$ such that $f(a) = 0$ and $f(b) = b^2$ is incorrect.

In conclusion, there are only two solutions for the statement equation: $f(x) = 0$ for all x, or $f(x) = x^2$ for all x.

OC334. Let p be an odd prime number. For positive integers k satisfying $1 \leq k \leq p-1$, the number of divisors of $kp+1$ between k and p exclusive is a_k. Find the value of $a_1 + a_2 + \cdots + a_{p-1}$.

Originally 2016 Japan Mathematical Olympiad Finals Problem 1.

We received 2 solutions of which 1 was correct and 1 was incomplete. We present the solution by Steven Chow slightly modified by the editor.

For each $1 \leq k \leq p-1$, let A_k be the set consisting of the divisors of $kp+1$ between k and p exclusive. Therefore for each k, $a_k = |A_k|$ and

$$a_1 + \cdots + a_{p-1} = |A_1| + \cdots + |A_{p-1}|\,.$$

We establish two facts about the sets $A_1, \ldots, A_{p-1}$.

First, $A_1, \ldots, A_{p-1}$ have no common elements, equivalently are mutually exclusive. Assume the opposite that there exists integers k_1, k_2, and j such that

$$1 \leq k_1 \leq p-1, 1 \leq k_2 \leq p-1, k_1 < k_2 < j < p$$

and j belongs to the intersection of A_{k_1} and A_{k_2}. It follows that j is a common divisor of k_1p+1 and k_2p+1, and consequently j is a divisor of the difference

$$(k_2p+1) - (k_1p+1) = (k_2 - k_1)\,p.$$

Since p is prime and $j < p$ we can conclude that j is a divisor of $k_2 - k_1$. However this implies $j \leq k_2 - k_1 < k_2 < j$, which is a contradiction. Therefore the sets $A_1, \ldots, A_{p-1}$ are mutually exclusive.

Second, the union of the sets $A_1, \ldots, A_{p-1}$ is $\{2, 3, \ldots, p-1\}$. Let j be an integer such that $2 \leq j \leq p-1$. Since j and p are relatively prime integers, there exist two integers k and m such that $kp = mj - 1$ and $1 \leq k \leq j-1$. A short argument for this fact is provided in the editor's comments at the end of the solution. Rearranging the last equation into $kp+1 = mj$, we find that j is a divisor of $kp+1$, and hence $j \in A_k$. Since j was arbitrary, the union $A_1 \cup \cdots \cup A_{p-1}$ is $\{2, 3, \ldots, p-1\}$, a set with $p-2$ elements.

In conclusion $p - 2 = |A_1 \cup \cdots \cup A_{p-1}| = |A_1| + \cdots + |A_{p-1}| = a_1 + \cdots + a_{p-1}$.

Editor's comments. Regardless of the value of p, we have $a_{p-1} = 0$ as there are no integers between $p-1$ and p. Therefore the question could have been asked with $1 \leq k \leq p-2$ and led to the conclusion that $p - 2 = a_1 + \cdots + a_{p-2}$.

The proof used the fact that if j and p are relatively prime integers then there exist two integers k and m such that $kp = mj - 1$ and $1 \leq k \leq j-1$. This can be obtained by looking at the remainders of $p, 2p, \ldots, (j-1)p$ when divided by j. There are $j-1$ such remainders, all different and taking only $j-1$ values: $1, \ldots, j-1$. Therefore there exists k such that $1 \leq k \leq j-1$ and the remainder of kp when divided by j is $j-1$. Equivalently, there exists k and m such that $1 \leq k \leq j-1$ and $kp = mj + j - 1$, or $kp = (m+1)j - 1$.

OC335. Medians AM_A, BM_B and CM_C of a triangle ABC intersect at M. Let Ω_A be the circumcircle of the triangle that passes through the midpoint of AM and is tangent to BC at M_A. Define Ω_B and Ω_C analogously. Prove that Ω_A, Ω_B and Ω_C intersect at one point.

Originally 2016 All Russian Olympiad Grade 11 Day 2 Problem 8.

We received 2 solutions of which 1 was correct and 1 was incomplete. We present the solution by Steven Chow slightly modified by the editor.

We use barycentric coordinates. Let $(1,0,0) = A$, $(0,1,0) = B$, and $(0,0,1) = C$. Let $a = BC$, $b = CA$, and $c = AB$. Let x, y, and z represent the first, second and third coordinate of an arbitrary point, respectively. Let u, v, and w be the numbers such that the equation of Ω_A is

$$0 = -a^2yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z).$$

Since $M_A = (0, 1/2, 1/2)$ is on Ω_A, $v + w = a^2/2$.

Since Ω_A is tangent to the line BC described by $x = 0$, the equation

$$0 = -a^2yz + (vy + wz)(y + z)$$

has unique solution y and z. This is equivalent to the quadratic equation

$$0 = v(y/z)^2 - (a^2/2)(y/z) + w$$

in y/z having unique solution, or the equation having zero discriminant. The zero discriminant implies that $vw = \left(a^2/4\right)^2$. Therefore $v = w = a^2/4$.

Since the midpoint of segment AM has barycentric coordinates $(4/6, 1/6, 1/6)$ and lies on Ω_A,

$$0 = -a^2 - 4b^2 - 4c^2 + \left(4u + a^2/2\right)(6),$$

which leads to $u = -a^2/12 + b^2/6 + c^2/6$.

Therefore the equation of Ω_A is

$$0 = -a^2yz - b^2zx - c^2xy + \left(\left(-\frac{a^2}{12} + \frac{b^2}{6} + \frac{c^2}{6}\right)x + \frac{a^2}{4}y + \frac{a^2}{4}z\right)(x + y + z). \quad (1)$$

Similarly, the equations of Ω_B and Ω_C are the cyclic forms of (1).

Next, we claim that the point Ω with barycentric coordinates in proportion

$$\left(\left(-2a^2+b^2+c^2\right)^2 : \left(a^2-2b^2+c^2\right)^2 : \left(a^2+b^2-2c^2\right)^2\right)$$

satisfies the equation of Ω_A, and hence lies on Ω_A. Similarly, by cyclicity, this point is also on Ω_B and Ω_C, and so is the intersection point of the three circles. The intersection point was identified by computing the radical centre of the three circles, which is the point of intersection of two of the three radical axes. The equation of a radical axis is computed by subtracting the equations of two circles.

To see that the point Ω satisfies the equation (1) of Ω_A, we introduce some notations $m=-2a^2+b^2+c^2$, $n=a^2-2b^2+c^2$, and $p=a^2+b^2-2c^2$. Note that $m+n+p=0$. Consequently, after taking the square of

$$m^2+n^2+p^2=-2(mn+np+pm),$$

we have:

$$\begin{aligned}\left(m^2+n^2+p^2\right)^2 &= 4(mn+np+pm)^2\\ &= 4\left(m^2n^2+n^2p^2+p^2m^2+2mnp\,(m+n+p)\right)\\ &= 4\left(m^2n^2+n^2p^2+p^2m^2\right). \qquad (2)\end{aligned}$$

Also, since $m=-(n+p)$,

$$m^2+n^2+p^2=(n+p)^2+n^2+p^2=2(n^2+p^2+np). \qquad (3)$$

The coordinates of Ω are in proportion $\left(m^2 : n^2 : p^2\right)$. We evaluate the second part of the equation (1) of Ω_A at $x_\Omega=m^2$, $y_\Omega=n^2$, and $z_\Omega=p^2$. We use (2), (3), and $m+n+p=0$ in the following computations:

$$\begin{aligned}&\left(\left(-\frac{a^2}{12}+\frac{b^2}{6}+\frac{c^2}{6}\right)x_\Omega+\frac{a^2}{4}y_\Omega+\frac{a^2}{4}z_\Omega\right)(x_\Omega+y_\Omega+z_\Omega)\\ &= \left(-\frac{a^2}{3}+\frac{b^2}{6}+\frac{c^2}{6}\right)x_\Omega(x_\Omega+y_\Omega+z_\Omega)+\frac{a^2}{4}(x_\Omega+y_\Omega+z_\Omega)^2\\ &= \frac{m}{6}m^2\left(m^2+n^2+p^2\right)+\frac{a^2}{4}\left(m^2+n^2+p^2\right)^2\\ &= \frac{m}{3}m^2\left(n^2+p^2+np\right)+a^2(m^2n^2+n^2p^2+p^2m^2)\\ &= a^2n^2p^2+\left(a^2+\frac{m}{3}\right)m^2p^2+\left(a^2+\frac{m}{3}\right)n^2m^2+m^2\frac{mnp}{3}\\ &= a^2n^2p^2+\left(b^2+\frac{n}{3}\right)m^2p^2+\left(c^2+\frac{p}{3}\right)n^2m^2+m^2\frac{mnp}{3}\\ &= a^2n^2p^2+b^2p^2m^2+c^2m^2n^2+\frac{m^2np}{3}(m+n+p)\\ &= a^2n^2p^2+b^2p^2m^2+c^2m^2n^2=a^2y_\Omega z_\Omega+b^2z_\Omega x_\Omega+c^2x_\Omega y_\Omega.\end{aligned}$$

Therefore Ω_A, Ω_B, and Ω_C intersect in one point.

FOCUS ON...

No. 32

Michel Bataille

Harmonic Ranges and Pencils

Introduction

Elementary properties of harmonic conjugacy can lead to simple and elegant solutions to some geometry problems. Before considering examples, let us recall the basic definitions. Let A, B, C, D be four distinct points on a line. We say that C, D are harmonic conjugates w.r.t A, B when C, D divide AB in the same ratio, that is, if $\frac{\overline{CA}}{\overline{CB}} = -\frac{\overline{DA}}{\overline{DB}}$ (here and in what follows, the bar indicates signed distance and w.r.t. means "with respect to"). Clearly, the latter is equivalent to $\frac{\overline{AD}}{\overline{AC}} = -\frac{\overline{BD}}{\overline{BC}}$, meaning that A, B are harmonic conjugates w.r.t. C, D. If either condition is satisfied, we say that A, B, C, D is a harmonic division or a harmonic range. Let I be the midpoint of AB. Starting with $\overrightarrow{AD} = -k\overrightarrow{AC}$ and $\overrightarrow{BD} = k\overrightarrow{BC}$ for some real number k, easy manipulations give $\overrightarrow{ID} = k\overrightarrow{IA}$ and $\overrightarrow{IA} = k\overrightarrow{IC}$ and conversely. Thus, the condition $IA^2 = \overline{IC}.\overline{ID}$ can also be used to prove the harmonicity of the range of collinear points A, B, C, D.

Harmonic pencil

Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four distinct lines which are either parallel or concurrent, and let transversals m, m' meet them in A, B, C, D and in A', B', C', D', respectively. If $\ell_1, \ell_2, \ell_3, \ell_3$ are parallel, then $\frac{\overline{C'A'}}{\overline{C'B'}} = \frac{\overline{CA}}{\overline{CB}}$ and $\frac{\overline{D'A'}}{\overline{D'B'}} = \frac{\overline{DA}}{\overline{DB}}$ so that A', B', C', D' is a harmonic range as soon as A, B, C, D is one (Figure 1).

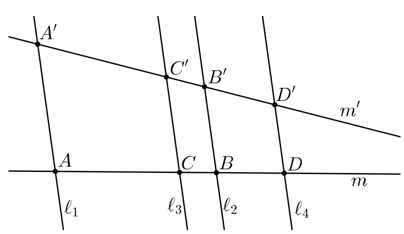


Figure 1

This conservation of harmonicity remains true when $\ell_1, \ell_2, \ell_3, \ell_4$ are concurrent lines. To prove this, we shall use the following lemma (for a proof, we refer the reader to **[1]** p. 169).

> Let A, B, C, D be four distinct points on a line and S a point not on this line. Let the parallel to SC through A intersect SD at M and SB at E. Then A, B, C, D is a harmonic range if and only if M is the midpoint of AE (Figure 2).

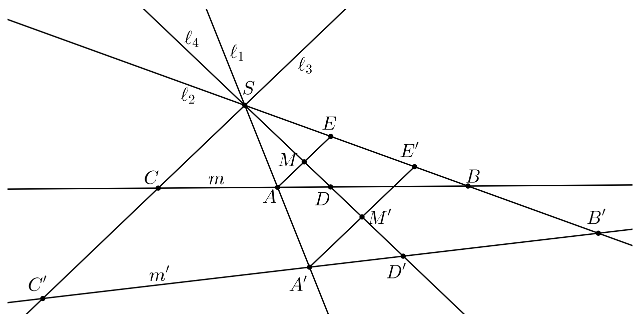


Figure 2

Now, let $\ell_1, \ell_2, \ell_3, \ell_4$ be concurrent at S and let m intersect them along a harmonic range A, B, C, D. If m' intersects them along A', B', C', D', we draw the parallels to SC through A and through A', which intersect SD and SB, respectively at M and E and at M' and E' (Figure 2). Since A, B, C, D is a harmonic range, M is the midpoint of AE; since AE is parallel to $A'E'$, M' is the midpoint of $A'E'$ and so A', B', C', D' is a harmonic range as well.

This justifies the following definition: $\ell_1, \ell_2, \ell_3, \ell_4$ is called a harmonic pencil when a transversal m intersects $\ell_1, \ell_2, \ell_3, \ell_4$ along a harmonic range. From the lemma above, an example is given by the lines ℓ, AM, AB, AC if M is the midpoint of the side BC of ΔABC and ℓ is the parallel to BC through A.

We are now ready to examine a few situations involving harmonic ranges or pencils and illustrate them with problems.

An angle and its bisectors

If ABC is a triangle (with $AB \neq AC$) and the internal and external bisectors of $\angle BAC$ meet BC at D and D', respectively, we know that D, D' divide BC in the ratio $\frac{AB}{AC}$. Thus, B, C, D, D' is a harmonic range and AB, AC, AD, AD' is a harmonic pencil. Note that AD, AD' are perpendicular. Interestingly, a kind of converse holds (easily proved or see **[1]** p. 170):

> Let $\ell_1, \ell_2, \ell_3, \ell_4$ be a harmonic pencil of concurrent lines at S. If ℓ_3, ℓ_4 are perpendicular, then they are the axes of symmetry of ℓ_1, ℓ_2.

To illustrate these results, we consider problem **3036** [2005 : 175 ; 2006 : 244], slightly modified:

> Let A, B, C be three distinct collinear fixed points. Let M be an arbitrary point not on the line ABC. The internal angle bisector of $\angle MAB$ intersects the line MB at a point X. The perpendicular at A to the line AX intersects the line MC at a point Y.
>
> (a) Prove that the line XY passes through a fixed point D.
>
> (b) Let Z be the projection of the point A onto the line XY. Prove that ZA is a bisector of $\angle BZC$.

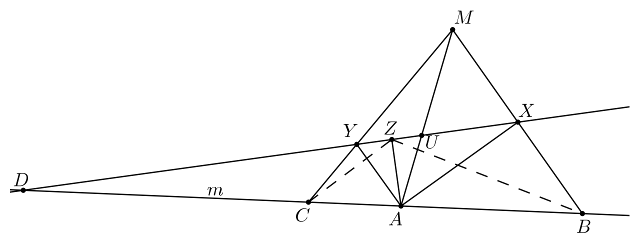


Figure 3

(a) Let XY intersect MA at U and the line m through A, B, C at D (Figure 3). Since AY and AX are the bisectors of $\angle MAB$, the pencil AY, AX, AM, AB is harmonic. Therefore, the range Y, X, U, D is harmonic and in consequence the pencil MY,MB,MU,MD is harmonic. Finally, considering the latter and the transversal m, we see that C, B, A, D is harmonic and conclude that XY always passes through the harmonic conjugate of A w.r.t. B, C.

(b) Since ZA and ZD are perpendicular and C, B, A, D is a harmonic range, ZA, ZD are the bisectors of $\angle BZC$.

About polars with respect to a circle

Consider a circle Γ with centre O and radius r and let M be a point distinct from O. The locus Π_M of points P such that the dot product $\overrightarrow{OM} \cdot \overrightarrow{OP}$ equals r^2 is the polar of M w.r.t. Γ. If M lies on Γ, M itself is a point of Π_M; otherwise, denoting by A and B the points of intersection of Γ and the line OM, we see that the harmonic conjugate M' of M w.r.t. A, B is a point of Π_M (since O is the midpoint of AB and $\overline{OM}.\overline{OM'} = OA^2$). Moreover, P is on Π_M if and only if $\overrightarrow{OM} \cdot \overrightarrow{OP} = \overrightarrow{OM} \cdot \overrightarrow{OM'}$, which is equivalent to $\overrightarrow{OM} \cdot \overrightarrow{M'P} = 0$, and therefore Π_M is the perpendicular to OM through M' (Figure 4). In the same way, we obtain that if M is on Γ, then Π_M is the tangent to Γ at M.

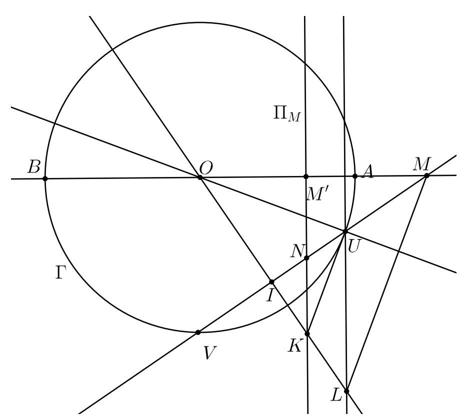


Figure 4

In passing, note that M' is the inverse of M in Γ and that M is on Π_N as soon as N is on Π_M (polar reciprocity).

The result that M, M', A, B is a harmonic range can be generalized as follows:

> If a line through M intersects Π_M at N and the circle Γ at U and V, then M, N, U, V is a harmonic range.

Let the perpendicular bisector of UV intersect UV at I, Π_M at K and the perpendicular to OM through U at L (Figure 4). Clearly, U is the orthocenter of ΔOML, hence OU is perpendicular to ML. But M is on Π_K (since K is on Π_M) and so UV is the polar of K. It follows that K is on the polar of U, that is, the tangent to Γ at U. Being both perpendicular to OU, ML and KU are parallel, and so are KN and LU. As a result, if $\overrightarrow{IU} = k\overrightarrow{IN}$, then $\overrightarrow{IL} = k\overrightarrow{IK}$ and so $\overrightarrow{IM} = k\overrightarrow{IU}$. The result follows.

As an application, we propose here a problem of the 50th Olympiad of Moldova [2009 : 377]:

> The quadrilateral $ABCD$ is inscribed in a circle. The tangents to the circle at A and C intersect at a point P not on BD and such that $PA^2 = PB.PD$. Prove that BD passes through the midpoint of AC.

Let Γ be the circumcircle of $ABCD$ and let O be its centre. The line PD intersects Γ again at B' with $B' \neq B$ (since P is not on BD). Since $PB' \cdot PD$ is the power of P w.r.t. Γ, we have $PB' \cdot PD = PA^2 = PB \cdot PD$, so that $PB' = PB$ and the line OP is the perpendicular bisector of BB' (Figure 5).

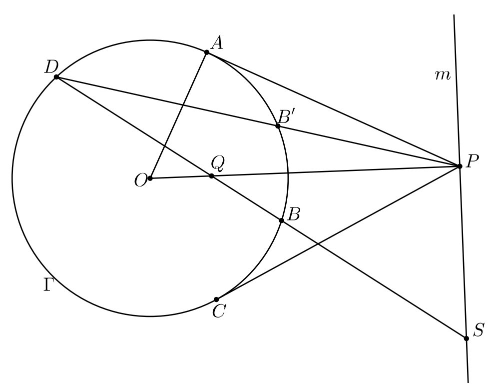


Figure 5

It follows that OP is a bisector of the angle $\angle BPD$ and so is the line m perpendicular to OP at P. As a result, PO, m, PD, PB is a harmonic pencil and the line BD intersects PO and m at Q and S such that Q, S, B, D is a harmonic range. From the property above, we then deduce that the polar of Q w.r.t. Γ passes through S, hence is m (since m is perpendicular to OQ). By polar reciprocity, Q is on the polar of P, which is AC, and the conclusion immediately follows.

Constructions with the straightedge alone

Harmonic ranges or pencils can be constructed with the straightedge alone. This interesting feature rests upon the following property:

> Let ℓ_1, ℓ_2 be two lines intersecting at S and A a point not on these lines. Through A we draw two transversals intersecting ℓ_1, ℓ_2 at M_1, M_2 and N_1, N_2. If the lines M_1N_2 and M_2N_1 intersect at U, then SA, SU, ℓ_1, ℓ_2 is a harmonic pencil.

The proof is easy: If B is the harmonic conjugate of A w.r.t. M_1, M_2, the line through A, N_1, N_2 is a transversal of the harmonic pencil UA, UB, UM_1, UM_2, hence intersects UB at C such that A, C, N_2, N_1 is a harmonic range. The line SB, which also passes through C, must coincide with SU.

Of course, if ℓ_1, ℓ_2 are parallel, a similar conclusion holds provided that SA and SU are replaced by the parallels to ℓ_1, ℓ_2 through A and U, respectively.

To see this at work, a good example is Problem **2965** [2004 : 367, 370; 2005 : 405]:

> Let $ABCD$ be a parallelogram. Using only an unmarked straightedge, find a point M on AB such that $AM = \frac{1}{5}AB$.

Here are the steps of the construction. First, we obtain the reflection B_1 of B in A by drawing ℓ such that DC, DA, DB, ℓ is a harmonic pencil (Figure 6a). The line ℓ intersects the line AB at B_1. Second, we construct B_1' such that A, B, B_1, B_1' is a harmonic range (Figure 6b). Finally, we repeat the first two steps with B_1' instead of B. This yields the desired point M as the harmonic conjugate w.r.t. A, B of the reflection B_2 of B_1' in A.

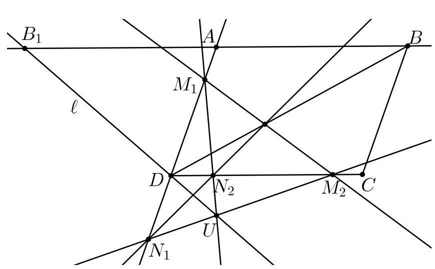


Figure 6a

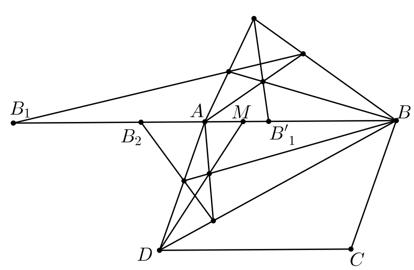


Figure 6b

Indeed, we have $\frac{\overline{B_1'A}}{\overline{B_1'B}} = -\frac{\overline{B_1A}}{\overline{B_1B}} = -\frac{1}{2}$ and so

$$\frac{\overline{AM}}{\overline{MB}} = \frac{\overline{B_2A}}{\overline{B_2B}} = -\frac{\overline{B_1'A}}{\overline{AB_1'} + \overline{AB}} = \frac{-\overline{B_1'A}/\overline{B_1'B}}{-\overline{B_1'A}/\overline{B_1'B} + \overline{AB}/\overline{B_1'B}} = \frac{1/2}{1/2 + 3/2} = \frac{1}{4}$$

The relation $AM = \frac{1}{5}AB$ follows.

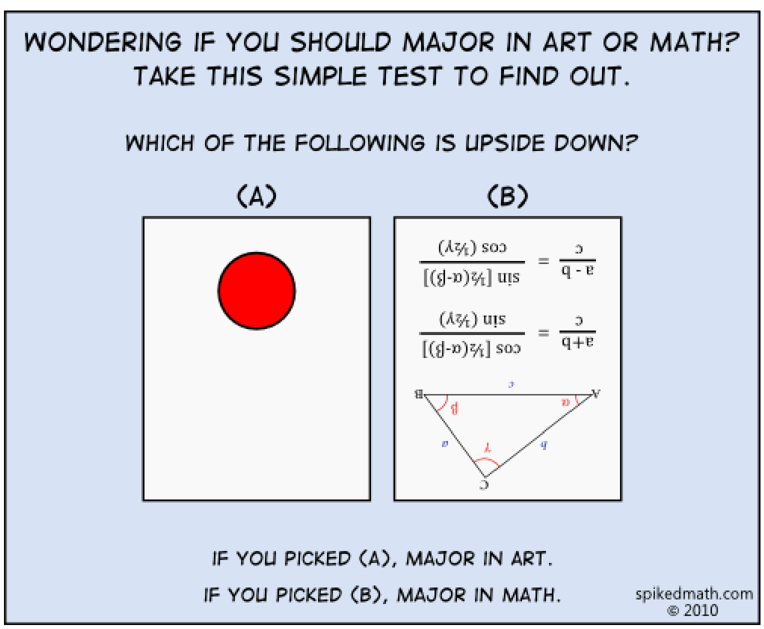
Exercises

1. Through a point P exterior to a given circle pass a secant and a tangent to the circle. The secant intersects the circle at A and B, and the tangent touches the circle at C on the same side of the diameter through P as A and B. The projection of C onto the diameter is Q. Prove that QC bisects $\angle AQB$. (Set at the competition Baltic Way in 2004.)

2. The standard construction for bisecting a line segment involves the use of two arcs and one straight line. Show that it can, in fact, be done with straight lines and just one arc. (Problem 88.I of the Mathematical Gazette, proposed in November 2004.)

Reference

[**1**] N. Altshiller-Court, *College Geometry*, Dover, 2007.



Linear Recurrence Sequences and Polynomial Division in Number Theory

Valcho Milchev

1 Introduction

The paper discusses some characteristics of linear recurrence sequences. It is inspired by the article "Polynomial Division in Number Theory" by James Rickards (***Crux***, 43(10), December 2017.).

Let us look at the following Diophantine equation which is symmetric in the unknowns x and y:

$$x^2 + pxy + y^2 = q, \tag{1}$$

where p and q are integers, and $p > 2$ (if $p \leq 2$, then there can only be a finite number of integer solutions). The integer solutions for $p > 2$ satisfy a linear recurrence equation

$$a_{n+2} = pa_{n+1} - a_n, \tag{2}$$

whose terms are integer coordinates on the hyperbola with equation (1).

Theorem 1 *Let $p > 2$ in the Diophantine equation. Then for any solution (x, y):*

(i) *if $q < 0$, x and y always have the same sign.*

(ii) *if $q > 0$, x and y, if they exist, can have the same or opposite signs.*

Theorem 2 *If the Diophantine equation (1) given $q \neq 0$ and $p > 2$ has solutions, then there is an infinite number of such solutions, which are pairs of neighbour terms of an integer sequence with linear recurrence equation (2).*

Proof. Let's assume that the equation (1) has a solution (x_1, y_1) in positive integers. By symmetry, we can assume without loss of generality that $x_1 \leq y_1$. Thus, considered as a quadratic in x, the equation $x^2 - pxy_1 + y_1^2 - q = 0$ has a root x_1. As $p > 2$, the discriminant $D = (p-4)\,y_1^2 + 4q > 0$, so there exists a second root x_2. By Vieta's formulas, we have $x_2 = py_1 - x_1$; as $p > 2$ and $x_1 \leq y_1$, we have $x_2 > y_1$.

Considering the equation $x_2^2 - px_2y + y^2 - q = 0$ as a quadratic in y, we find that a positive integer y_2 exists so that $y_2 > x_2$ and $x_2^2 - p_2x_2 + y_2^2 - q = 0$. Continuing this process, we obtain an increasing sequence of natural numbers. Every three neighbour terms respectively fulfil Vieta's formula. The process is known as "Vieta jumping", and yields an endless linear recurrence sequence of integers with a recurrence equation (2). Analogously, we can develop a jumping "downwards". Every two neighbour terms of the resulting sequence $\{a_n\}$ satisfy the Diophantine equation (1). □

Figure 1 shows the two branches of the hyperbola $x^2 - 3xy + y^2 = 1$ (an example with $q > 0$). The integer points on its graph have coordinates that are pairs of consecutive terms of the sequence $\{\ldots, -8, -3, -1, 0, 1, 3, 8, \ldots\}$ satisfying $a_{n+2} = 3a_{n+1} - a_n$. Figure 2 shows an example with $q < 0$, the hyperbola $x^2 - 3xy + y^2 = -1$. The integer points have coordinates that are pairs of consecutive terms of the sequence $\{\ldots, 5, 2, 1, 1, 2, 5, \ldots\}$, which also satisfies $a_{n+2} = 3a_{n+1} - a_n$.

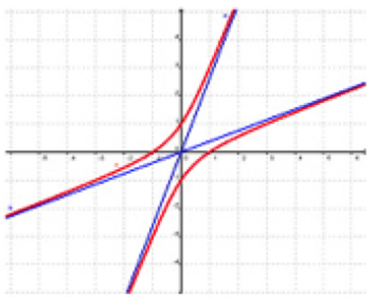


Figure 1

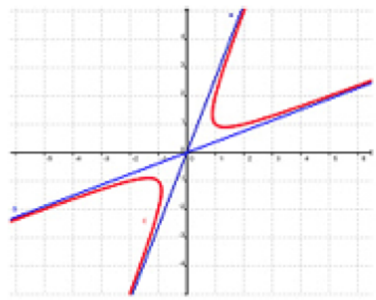


Figure 2

Theorem 3 *If the Diophantine equation* (1) *with* $p > 2$, $q > 0$ *has integer solutions, then* q *is a perfect square if and only if there is an integer point on the hyperbola, one of whose coordinates is equal to zero.*

Criterion 1 *If the Diophantine equation* (1) *with* $p > 2$, $q > 0$, *and* $q \leq p$ *has a solution, then* q *is the perfect square of an integer.*

Below, we will apply these results to some contest problems.

Problem 1 (1988 International Mathematical Olympiad) *Let* a *and* b *be such positive integers that* $ab + 1$ *divides* $a^2 + b^2$. *Prove that the number* $\dfrac{a^2 + b^2}{ab + 1}$ *is a perfect square.*

(The problem was considered the hardest at the 29th IMO. It is said that five number theorists from Australia were not able to solve the problem in five hours! With modern techniques, this problem does not look that tough.)

Solution. We must solve the Diophantine equation $a^2 - pab + b^2 = q$ for $p = q = 1$. According to the formulated criterion, if this equation has a solution in integers, then p is a perfect square of an integer. All solutions are given by a linear recurrence sequence which includes the number zero.

Let $p = m^2$. Since we have already concluded that zero is part of the sequence, then by "Vieta Jumping" the pair $(0, m)$ yields a sequence of polynomials $0, m, m^2, m^3, m^5 - m, m^7 - 2m^3, \ldots$, obtained through the recurrence relation $a_{n+1} = m^2 a_n - a_{n-1}$. This sequence of polynomials provides all solutions of the problem. The hyperbola is of the type illustrated in Figure 1, and both branches cross the coordinate axis in integer points. $\square$

Problem 2 (1998 Canadian Mathematical Olympiad, National Round)

Let the sequence $\{a_n\}$ be defined by $a_0 = 0$, $a_1 = m$, $a_{n+1} = m^2 a_n - a_{n-1}$. Prove that all integer solutions (a, b) of the equation $\dfrac{a^2 + b^2}{ab + 1} = m$ given $a < b$ coincide with the pairs (a_n, a_{n+1}).

Problem 3 (Kvant magazine, M1225) *Prove that if for the natural numbers a and b the number $\dfrac{a^2 + b^2}{ab - 1}$ is also natural, then $\dfrac{a^2 + b^2}{ab - 1} = 5$.*

Solution. Denote $\frac{a^2+b^2}{ab-1} = p$. Then $a^2 - pab - b^2 = -p$. As $a \neq b$ and $b > 0$, we reach a version of (1) with $q = -p < 0$. In this case, the hyperbola does not cross the coordinate axis, and the respective sequences of solutions, if such, include only positive or negative terms. Let's look at the sequence with positive numbers; let y_0 be its least element. The equation $x^2 - pxy_0 + y^2 = -p$ has integer solutions x_1 and x_2, according to the assumption. Let $y_0 < x_1 \leq x_2$, which means

$$\frac{1}{2}\left(py_0 - \sqrt{(p^2 - 4)\, y_0^2 - 4p}\right) > y_0.$$

Thus $p > 2$, $(p-2)y_0^2 < p$. As p and y_0 are natural numbers, $y_0 = 1$. The discriminant becomes $p^2 - 4 - 4p$; as it is a perfect square, and differs by 8 from the perfect square $p^2 + 4p - 4$, we must have $p = 5$. This way we find $x_1 = 2$, $x_2 = 3$ and deduce that the Diophantine equation $x^2 - 5xy + y^2 + 5 = 0$ has an infinite number of solutions in natural numbers. These are adjacent pairs of terms from the sequence $\{\ldots, 14, 3, 1, 2, 9, \ldots\}$ with linear recurrence equation $a_{i+2} = 5a_{i+1} - a_i$. They are also the pairs of coordinates of the integer points on the hyperbola $x^2 - 5xy + y^2 + 5 = 0$. □

Problem 4 (2007 Spain Mathematical Olympiad, National Round) *Find all the possible positive integers which the expression $\dfrac{m^2 + mn + n^2}{mn - 1}$ can take where m and n are natural numbers and $mn \neq 1$.*

Problem 5 (2013 British Mathematical Olympiad) *Find all pairs of natural numbers x and y for which x divides $y^2 + 1$ and y divides $x^2 + 1$.*

Problem 6 (2012 Vietnam Mathematical Olympiad) *Let a and b be two odd natural numbers, where a is a divisor of $b^2 + 2$, and b is a divisor of $a^2 + 2$. Prove that a and b are terms of the sequence of natural numbers $\{v_n\}$ for which $v_1 = v_2 = 1$, $v_n = 4v_{n-1} - v_{n-2}$ if $n \geq 3$.*

Problem 7 (1999 Bulgaria Team Selection Test) *Prove that the number $n^4 + 1$ has a divisor of the $nm - 1$ type (m and n are natural numbers) if and only if n is a term of the sequence $\{a_i\}_{i=1}^n$ for which $a_1 = 1$, $a_2 = 2$ or $a_2 = 3$, $a_{i+2} = 5a_{i+1} - a_i$.*

Comment. This problem is based on the fact that the defined sequence $\{a_i\}_{i=1}^n$ has the property $a_k^4 + 1 = (a_{k-1}a_k - 1)(a_k a_{k+1} - 1)$. This property is peculiar to the specific sequence of this problem; it is not shared by all sequences with the linear recurrence equation $a_{i+2} = 5a_{i+1} - a_i$.

We finish with a pair of harder problems (with solutions).

Problem 8 (2007 International Mathematical Olympiad) *Let a and b be natural numbers for which $4ab-1$ divides $\left(4a^2-1\right)^2$. Prove that $a=b$.*

Solution. In this problem the relation $(a-b)^2 = k(4ab-1)$ is valid where $k \geq 0$ is an integer. This follows from the equation

$$\left(4a^2-1\right)^2 = \left(4a^2-4ab+4ab-1\right)^2 = 16a^2(a-b)^2 + (4ab-1)(8a^2-4ab-1),$$

when considered that $4ab-1$ and $16a^2$ are mutually prime. When $k=0$ we have $a=b$. Let's assume the possibility that $k \geq 1$, i.e. $a \neq b$. If integer solutions (a,b) exist, then they are pairs of consecutive terms of the linear recurrence sequence obtained through "Vieta Jumping". Let b_0 be the least positive integer in this sequence. From Vieta's formulas we can deduce that positive integers a_1 and a_2 exist so that the pairs (b_0, a_1) and (b_0, a_2) are solutions to the equation in this case. Moreover, $a_1 > b_0$ and $a_2 > b_0$, which means that $(a_1-1)(a_2-1) \geq b_0^2$, and so we obtain

$$b_0^2 + k - 2(2k+1)b_0 = a_1a_2 - (a_1+a_2) = (a_1-1)(a_2-1) - 1 \geq b_0^2 - 1,$$

whence $b_0 \leq \dfrac{k+1}{4k+2} \leq 1$, a contradiction. $\square$

Problem 9 (La Gaceta de la RSME, Vol. 17 (2014), Problema 241) *Find all positive integers a and b for which the expression $\dfrac{a^4-a^2+1}{ab-1}$ is a positive integer.*

Solution. If $a=1$, then $b=2$. Let $a>1$, and let a and b fulfil the condition of the problem. We can write $a^4-a^2+1 = (ab-1)\,B$; then $a^4-a^2 = abB-(B+1)$. Consequently, a divides $B+1$, so $a^4-a^2+1 = (ab-1)\,(ka-1)$. We rewrite the numerator:

$$\begin{aligned} a^4-a^2+1 &= \left(b^4-b^2+1\right)a^4 - \left(a^2b^2-1\right)\left(a^2b^2-a^2+1\right) \\ &= \left(b^4-b^2+1\right)a^4 - (ab-1)A, \end{aligned}$$

where A is a positive integer. As a^2 and $ab-1$ are mutually prime, b^4-b^2+1 is divisible by $ab-1$. In the same way, we find that k^4-k^2+1 is divisible by $ka-1$. Obviously $a \neq b$ and $a \neq k$. If $a<b$, then $k<a$, because otherwise we would find that $a^4-a^2+1 = (ab-1)(ka-1) \geq \left(a^2+a-1\right)^2$, which is impossible. Therefore

$$ab(ka-1) = a^4-a^2+1+(ka-1) \quad \text{or} \quad b = \frac{a^2+k^2-1}{ka-1}a - k = pa-k.$$

Then p is also a positive integer. We obtain the equation

$$a^2 - pka + k^2 + p - 1 - 0,$$

which is symmetric in relation to a and k (and has no solutions for $p \leq 2$). Let (a_0, k_0) be the integer solution minimizing the value of k_0 (note that $k_0 \leq a_0$), and let a_1 be the other root of

$$a^2 - pk_0a + k_0^2 + p - 1 - 0.$$

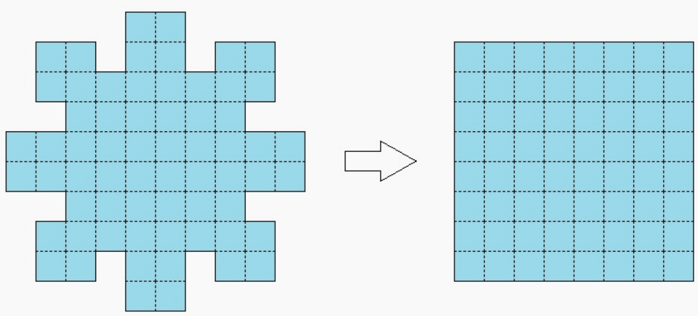
We can consider $a_1 \geq a_0 > k_0$. The root a_1 also is a positive integer due to the relation $a_1 = pk_0 - a_0$. Then

$$k_0 < a_0 = \frac{pk_0 - \sqrt{p^2k_0^2 - 4k_0^2 - 4p + 4}}{2}.$$

The only possibility is $k_0 = 1$, from where it follows that $p = 4$. This way we find $a_0 = 2$ and $a_1 = 2$. Because a, b, and k are symmetrical, we conclude that the solutions a, b of the problem are consecutive terms of the sequence $\{1, 2, 7, 26, \dots\}$ with linear recurrence equation $x_{n+2} = 4x_{n+1} - x_n$. $\square$

Quadrature of the figure

Take a grid paper and cut out the figure shown below on the left. Can you cut it into 5 pieces and arrange them to form an 8×8 square?



Puzzle by Nikolai Avilov.

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **February 1, 2019**.

The editor thanks Valérie Lapointe, Carignan, QC, for translations of the problems.

An asterisk ($\star$) *after a number indicates that a problem was proposed without a solution.*

4361. *Proposed by Andrew Wu.*

Let ABC be a scalene triangle with circumcircle Γ, circumcenter O and incenter I. Suppose that L is the midpoint of the arc BAC of Γ. The perpendicular bisector of AI meets at X the arc AC that contains B, and at Y the arc AB that contains C. Let XL and AC meet at P; let YL and AB meet at Q. Show that the orthocenter of triangle OPQ lies on XY.

4362. *Proposed by Oai Thanh Dao and Leonard Giugiuc.*

Let $ABCD$ be a convex quadrilateral and let F be the midpoint of CD. Consider a point E inside $ABCD$ such that $AE \cdot CE = BE \cdot DE$. The lines EF and AB intersect at G. If $\angle AED + \angle CEB = 180°$, prove that $\angle AED = \angle AGE$.

4363. *Proposed by Michel Bataille.*

Let $(a_n)_{n\geq 0}$ be the sequence defined by $a_0 > 0$ and the recursion

$$a_{n+1} = \frac{a_n}{1+(n+1)a_n^2}.$$

Prove that the series $\sum\limits_{n=0}^{\infty} a_n^2$ is convergent and find $\lim\limits_{n\to\infty}\left(n \cdot \sum\limits_{k=n}^{\infty} a_k^2\right)$.

4364. *Proposed by George Stoica.*

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a binary operation, and define $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$g(a,b) = \begin{cases} a \text{ if } b = 1 \\ f(g(a,b-1),a) \text{ if } b \geq 2. \end{cases}$$

If f is associative and g is commutative, prove that $f(a,b) = a+b$ and $g(a,b) = ab$.

4365. *Proposed by Marius Drăgan and Neculai Stanciu.*

Let a and b be real numbers such that $a+b, a^4$ and b^4 are rational numbers and $a+b \neq 0$. Prove that a ad b are rational numbers.

4366. *Proposed by Daniel Sitaru.*

Let x_n be the base angle of a right triangle with base n and altitude 1. Find

$$\sum_{k=1}^{\infty} x_{k^2+k+1}.$$

4367. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Let a, b and c be distinct complex numbers such that $|a| = |b| = |c| = 1$ and $|a+b+c| \leq 1$. Prove that

$$\left|\frac{(a+b)(b+c)}{(a-b)(b-c)}\right| + \left|\frac{(b+c)(c+a)}{(b-c)(c-a)}\right| + \left|\frac{(c+a)(a+b)}{(c-a)(a-b)}\right| = 1.$$

4368. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate

$$\sum_{n=2}^{\infty} \left[2^n \left(\zeta(n) - 1\right) - 1\right],$$

where ζ denotes the Riemann zeta function defined as $\zeta(z) = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^z}$.

4369. *Proposed by Mihaela Berindeanu.*

On the sides of $\triangle\, ABC$ take points $A_1,\ A_2 \in (BC),\ B_1,\ B_2 \in (AC),\ C_1,\ C_2 \in (AB)$, so that $BA_1 = A_2C,\ CB_1 = B_2A,\ AC_1 = C_2B$. On $B_2C_1,\ A_1C_2,\ A_2B_1$ take $A_3,\ B_3,\ C_3$ so that $\dfrac{C_1A_3}{A_3B_2} = \dfrac{A_1B_3}{B_3C_2} = \dfrac{B_1C_3}{C_3A_2} = k$. Find all values of k for which $AA_3,\ BB_3,\ CC_3$ are concurrent lines.

4370. *Proposed by Leonard Giugiuc and Sladjan Stankovik.*

Solve the following system of equations:

$$\begin{cases} a+b+c+d = 4, \\ a^2+b^2+c^2+d^2 = 7, \\ abc+abd+acd+bcd-abcd = \frac{15}{16}. \end{cases}$$

. . . .

4361. *Proposé par Andrew Wu.*

Soit ABC un triangle scalène et le cercle circonscrit Γ de centre O et soit le point I, le centre du cercle inscrit. Soit L le point milieu de l'arc BAC de Γ. La médiatrice du segment AI rencontre au point X l'arc AC qui contient B et rencontre au point Y l'arc AB qui contient C. Soit P, le point d'intersection de XL et AC et soit Q, le point d'intersection de YL et AB. Montrez que l'orthocentre du triangle OPQ est sur le segment XY.

4362. *Proposé par Oai Thanh Dao et Leonard Giugiuc.*

Soit $ABCD$ un quadrilatère convexe et soit F le point milieu du segment CD. Considérez un point E dans $ABCD$ tel que $AE \cdot CE = BE \cdot DE$. Les segments EF et AB s'interceptent au point G. Si $\angle AED + \angle CEB = 180°$, prouvez que $\angle AED = \angle AGE$.

4363. *Proposé par Michel Bataille.*

Soit $(a_n)_{n\geq 0}$ une suite définie par $a_0 > 0$ et la récurrence

$$a_{n+1} = \frac{a_n}{1+(n+1)a_n^2}.$$

Prouvez que la série $\sum\limits_{n=0}^{\infty} a_n^2$ est convergente et trouvez $\lim\limits_{n\to\infty}\left(n \cdot \sum\limits_{k=n}^{\infty} a_k^2\right)$.

4364. *Proposé par George Stoica.*

Soit $f : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ une opération binaire et soit $g : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ définie telle que

$$g(a,b) = \begin{cases} a \text{ si } b = 1 \\ f(g(a,b-1),a) \text{ si } b \geq 2. \end{cases}$$

Si f est associative et g est commutative, prouvez que $f(a,b) = a+b$ et $g(a,b) = ab$.

4365. *Proposé par Marius Drăgan et Neculai Stanciu.*

Soit a et b des nombres réels tels que $a+b, a^4$ et b^4 sont des nombres rationnels et $a+b \neq 0$. Prouvez que a et b sont des nombres rationnels.

4366. *Proposé par Daniel Sitaru.*

Soit x_n l'angle à la base d'un triangle rectangle de base n et de hauteur 1. Trouvez

$$\sum_{k=1}^{\infty} x_{k^2+k+1}.$$

4367. *Proposé par Kadir Altintas et Leonard Giugiuc.*

Soit a, b et c des nombres complexes distincts tels que $|a| = |b| = |c| = 1$ et $|a+b+c| \leq 1$. Prouvez que

$$\left|\frac{(a+b)(b+c)}{(a-b)(b-c)}\right| + \left|\frac{(b+c)(c+a)}{(b-c)(c-a)}\right| + \left|\frac{(c+a)(a+b)}{(c-a)(a-b)}\right| = 1.$$

4368. *Proposé par Ovidiu Furdui et Alina Sîntămărian.*

Calculez

$$\sum_{n=2}^{\infty} \left[2^n \left(\zeta(n) - 1\right) - 1\right],$$

où ζ indique la fonction zeta de Riemann définie par $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$.

4369. *Proposé par Mihaela Berindeanu.*

Sur le triangle ABC, on prend les points $A_1,\ A_2 \in (BC)\,,\ B_1,\ B_2 \in (AC)\,,\ C_1,\ C_2 \in (AB)$, tel que $BA_1 = A_2C,\ CB_1 = B_2A,\ AC_1 = C_2B$. Sur les segments $B_2C_1,\ A_1C_2,\ A_2B_1$ on prend les points $A_3,\ B_3,\ C_3$ tels que

$$\frac{C_1A_3}{A_3B_2} = \frac{A_1B_3}{B_3C_2} = \frac{B_1C_3}{C_3A_2} = k.$$

Trouvez la valeur de k pour laquelle $AA_3,\ BB_3,\ CC_3$ sont des droites concourantes.

4370. *Proposé par Leonard Giugiuc et Sladjan Stankovik.*

Résolvez le système d'équations suivant :

$$\begin{cases} a+b+c+d = 4, \\ a^2+b^2+c^2+d^2 = 7, \\ abc+abd+acd+bcd-abcd = \frac{15}{16}. \end{cases}$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2017: 43(7), p. 302–306.

4261. *Proposed by Margarita Maksakova.*

Consider the chess board. A baron can move only on the black squares and in one move he can go from one black square to any of the diagonally adjacent black squares. What is the smallest number of moves he needs to go to every black square?

We received no correct and complete solution, so the problem remains open. One submission showed that it is possible to go through every square in 34 moves. Can you show whether this is the smallest number of moves needed?

4262. *Proposed by Prithwijit De.*

Let $a_1, a_2, \ldots, a_n$ be positive integers and suppose $\sum\limits_{k=1}^{n} a_k = S$. Find the smallest positive value of c such that the equation

$$\sum_{k=1}^{n} \frac{a_k x^k}{1+x^{2k}} = c$$

has a unique real solution.

We received 6 correct solutions and one incomplete submission. We present the solution of Ivko Dimitrić, slightly modified by the editor.

Let $L(x)$ denote the left hand side of the equation. Then $L(0) = 0$. Suppose $x \neq 0$. Then

$$L(x) = \sum_{k=1}^{n} \frac{a_k x^k}{1+x^{2k}} = \sum_{k=1}^{n} \frac{a_k}{x^k + \frac{1}{x^k}}.$$

Note that $L(x) = L(1/x)$. We have

$$\begin{aligned} L(x) = \sum_{k=1}^{n} \frac{a_k}{x^k + \frac{1}{x^k}} &\leq \frac{1}{2} \sum_{k=1}^{n} a_k \frac{2}{|x^k + \frac{1}{x^k}|} = \frac{1}{2} \sum_{k=1}^{n} a_k \frac{2}{|x|^k + \frac{1}{|x|^k}} \\ &\leq \frac{1}{2} \sum_{k=1}^{n} a_k \sqrt{|x|^k \cdot \frac{1}{|x|^k}} = \frac{1}{2} \sum_{k=1}^{n} a_k = \frac{S}{2}, \end{aligned}$$

where the last inequality is the AM-GM inequality. Note that equality $L(x) = \frac{S}{2}$ is satisfied if and only if $x > 0$ and $x^k = 1/x^k$ for all k, implying $x = 1$ is a unique solution for $L(x) = S/2$. Now suppose $0 < c < S/2$. By the intermediate value theorem and $L(0) = 0$, there exists $0 < r < 1$ with $L(r) = c$. But since $L(1/r) = L(r)$, the equation $L(x) = c$ does not have a unique solution. Thus $c = S/2$.

4263. *Proposed by Michel Bataille.*

Let ABC be a triangle. Let Γ, with centre O and radius R, be the circumcircle of ABC and γ, with centre $I \neq O$ and radius r, be the incircle of ABC. Let D, E, F be the orthogonal projections of the inverse of I in Γ onto BC, CA, AB, respectively. Express the circumradius of ΔDEF as a function of R and r.

We received 3 submissions, all of which were correct; we feature the solution by Andrew David Ionascu, slightly modified by the editor.

We denote the inverse of I with respect to Γ by J (that is, $OJ \times OI = R^2$), and denote by D', E', and F' the points of tangency where the incircle touches the respective sides BC, CA, and AB. Note that because $\frac{OJ}{OA} = \frac{OA}{OI}$, the triangles JAO and AIO are similar by side-angle-side. Therefore, $\frac{JA}{AI} = \frac{AO}{IO} = \frac{R}{IO}$, which by Euler's formula (namely $OI^2 = R^2 - 2Rr$) becomes

$$\frac{JA}{AI} = \frac{R}{\sqrt{R^2 - 2Rr}}.$$

Because of the right angles at E and F, the circle whose diameter is JA contains E and F and is therefore the circumcircle of ΔEAF. The Law of Sines applied to this triangle gives us

$$\frac{EF}{\sin \angle EAF} = JA. \tag{1}$$

Similarly, because of right angles at E' and F', the circle whose diameter is IA contains E' and F' and is therefore the circumcircle of $\Delta E'AF'$, whence

$$\frac{E'F'}{\sin \angle E'AF'} = IA. \tag{2}$$

Because $E' \in AE$ and $F' \in AF$, the angles $\angle EAF$ and $E'AF'$ are equal or supplementary, and division using equations (1) and (2) yields

$$\frac{EF}{E'F'} = \frac{JA}{AI} = \frac{R}{\sqrt{R^2 - 2Rr}}.$$

In the same way, we can show that

$$\frac{FD}{F'D'} = \frac{DE}{D'E'} = \frac{R}{\sqrt{R^2 - 2Rr}}.$$

Thus triangles DEF and $D'E'F'$ are similar (by side-side-side), so that if x is the circumradius of ΔDEF (which we seek) while the inradius r is the circumradius of $\Delta D'E'F'$, we must also have $\frac{x}{r} = \frac{R}{\sqrt{R^2-2Rr}}$. We conclude that the circumradius of ΔDEF satisfies

$$x = \frac{Rr}{\sqrt{R^2 - 2Rr}}.$$

Editor's comments. The proposer, whose key step was the same as that of our featured solution, observed that the result can be found as a theorem in Nathan Altshiller-Court, *College Geometry*, Dover, 1980, paragraph 362, p. 173: *The pedal triangles of two points for a given triangle are similar if and only if the two points are inverse with respect to the circumcircle of the given triangle.*

4264. *Proposed by Dorin Marghidanu and Leonard Giugiuc.*

Let (a_n) and (b_n) be two sequences such that $a_0, b_0 > 0$ and

$$a_{n+1} = a_n + \frac{1}{2b_n} \quad \text{and} \quad b_{n+1} = b_n + \frac{1}{2a_n}$$

for all $n \geq 0$. Prove that

$$\max(a_{2017}, b_{2017}) > 44.$$

We received 11 correct solutions. We present here the solution by Paolo Perfetti.

The quantity

$$f(a_n, b_n) = \frac{a_n}{b_n}$$

is invariant under the recurrence. Indeed,

$$f(a_{n+1}, b_{n+1}) = \frac{a_n + \frac{1}{2b_n}}{b_n + \frac{1}{2a_n}} = \frac{a_n}{b_n} = f(a_n, b_n)$$

This means that

$$\frac{a_n}{b_n} = f(a_n, b_n) = f(a_0, b_0) = \frac{a_0}{b_0}$$

and then

$$a_{n+1} = a_n + \frac{1}{2a_n}\frac{a_0}{b_0} \quad \text{and} \quad b_{n+1} = b_n + \frac{1}{2b_n}\frac{b_0}{a_0}.$$

Thus,

$$a_{n+1}^2 + b_{n+1}^2 = a_n^2 + b_n^2 + \underbrace{\left(\frac{a_0}{b_0} + \frac{b_0}{a_0}\right)}_{\geq 2\,(AGM)} + \underbrace{\left(\frac{1}{4a_n^2}\frac{a_0^2}{b_0^2} + \frac{1}{4b_n^2}\frac{b_0^2}{a_0^2}\right)}_{\geq 0} \geq \; a_n^2 + b_n^2 + 2,$$

and

$$a_{n+1}^2 + b_{n+1}^2 > 2(n+1) + a_0^2 + b_0^2 > 2(n+1).$$

Moreover,

$$\max(a_{2017}, b_{2017}) \geq \frac{\sqrt{a_{2017}^2 + b_{2017}^2}}{\sqrt{2}} > \sqrt{2017} > 44.9.$$

4265. *Proposed by Daniel Sitaru.*

Consider real numbers $a, b, c \in (0,1)$ such that $a+b+c=1$. Show that

$$\frac{4}{\pi}(\arctan a + \arctan b + \arctan c) > \frac{1}{2-(ab+bc+ca)}.$$

There were 9 correct solutions and 2 incorrect submissions submitted. We follow the independent solutions of Daniel Dan; and the team D. Bailey, E. Campbell, and C. Diminnie.

Since $(4/\pi)\arctan x$ is concave for $x \geq 0$ and is equal to x for $x=0$ and $x=1$,

$$\frac{4}{\pi}\arctan x \geq x$$

for $0 \leq x \leq 1$. Therefore the left side of the inequality is not less than $a+b+c=1$.

Since

$$\begin{aligned} 2(ab+bc+ca) &= (a+b+c)^2 - (a^2+b^2+c^2) \\ &= 1-(a^2+b^2+c^2) \\ &\leq 1-(ab+bc+ca), \end{aligned}$$

then $ab+bc+ca \leq 1/3$ and

$$\frac{1}{2-(ab+bc+ca)} \leq \frac{3}{5} < 1.$$

The result follows.

Editor's comments. Some solvers used $\arctan x \leq x-(x^3/3)$ and standard inequalities to get the lower bound $8/3\pi$ for the left side. One solver used Karamata's inequality for the concave function $\arctan x$ and the triples (a,b,c), $(1,0,0)$ to show that the left side was not less than $4/\pi$.

4266. *Proposed by Marius Stănean.*

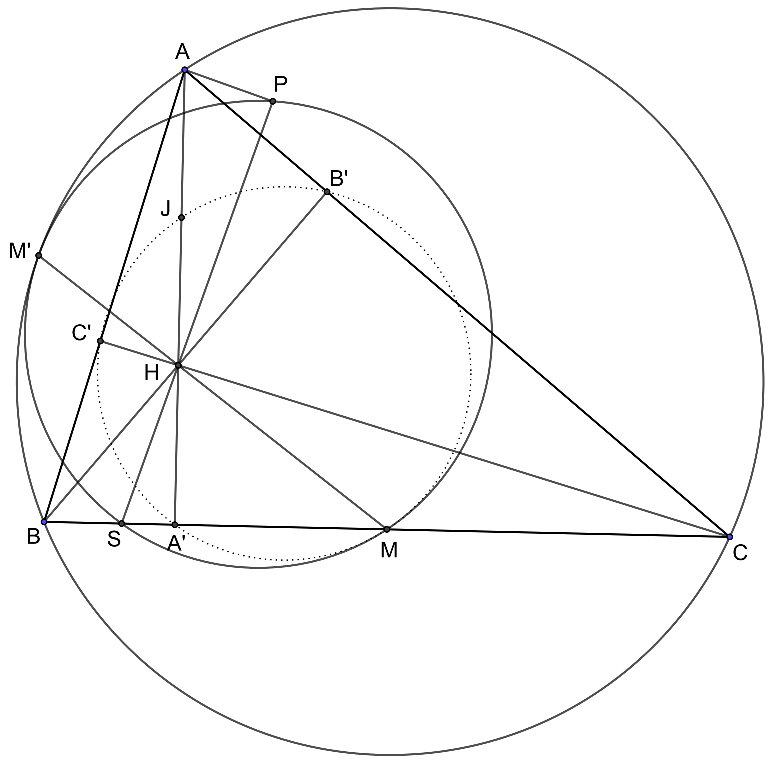
Let ABC be a triangle with orthocenter H. Let HM be the median and HS be the symmedian in triangle BHC. Denote by P the orthogonal projection of A onto HS. Prove that the circumcircle of triangle MPS is tangent to the circumcircle of triangle ABC.

We received 3 solutions. We present the solution by Michel Bataille modified by the editor.

To ensure the existence of $\triangle BHC$, we assume that ΔABC is not right-angled at B or C. Consider the case when $\triangle ABC$ is right-angled at A; then A, H and P coincide. The circumcentre of $\triangle ABC$ is M. Moreover, one can show that S coincides with the foot of the altitude from A, hence the circumcentre of

$\triangle PSM = \triangle ASM$ is the midpoint of AM. The required result follows in this case.

From now on, we suppose that ΔABC is not right-angled. Below is included a diagram for the case when $\triangle ABC$ is acute (the argument will also work when $\triangle ABC$ is obtuse, but points might be in a different order on lines and circles).



Let A', B', C' be the feet of the altitudes of $\triangle ABC$ (with A' on BC, and so on), and J be the midpoint of the line segment AH. Denote by $\mathbf{I}$ the inversion with centre H such that $\mathbf{I}(B) = B'$. Consider the effect of this inversion on the two circles in which we are interested.

Look at $\bigcirc ABC$, the circumcircle of $\triangle ABC$. Since $\angle BB'C = \angle CC'B = 90°$, the points B, B', C and C' are concyclic. The power of the point H gives us $HB' \cdot HB = HC' \cdot HC$, whence $\mathbf{I}(C) = C'$. Similarly, $\mathbf{I}(A) = A'$. Hence the image of the circumcircle of $\triangle ABC$ under $\mathbf{I}$ is the Euler circle $\bigcirc A'B'C'$ (a.k.a. the 9-point circle, which is known to also go through M and J, and which is shown dotted in the diagram).

Now consider the circumcircle of $\triangle PSM$. Similar to the above, we have $\angle APS = \angle AA'S = 90°$, so the points A, A', P and S are concyclic and the power of the point H gives us $HA' \cdot HA = HP \cdot HS$. Recall that $\mathbf{I}(A) = A'$, so it follows that $\mathbf{I}(P) = S$. Since the two points P and S on $\bigcirc PSM$ get mapped to each other, we conclude that the image of $\bigcirc PSM$ is itself.

It follows that, in order to prove that $\bigcirc ABC$ and $\bigcirc PSM$ are tangent, it suffices to show that $\bigcirc A'B'C'$ and $\bigcirc PSM$ are tangent. To this end, we will show that the centre of the Euler circle, the centre of $\bigcirc PSM$, and the point M are collinear. Denote $\mathbf{I}(M)$ by M'; note that M' is on both $\bigcirc ABC$ and $\bigcirc PSM$.

From earlier, $\mathbf{I}(S) = P$ and $\mathbf{I}(A') = A$; it follows that the line through SA' gets mapped to the circumcircle of $\triangle PAH$ (recall that an inversion with center H will map lines to circles that go through H). Thus M', B' and C', which are images of points on the line segment BC, are all on $\bigcirc PAH$. From the given setup of the problem, $\angle BHS = \angle CHM$, so $\angle M'HC' = \angle PHB'$; thus the arcs $M'C'$ and PB' on $\bigcirc PAH$ are congruent, which implies $M'P \parallel C'B'$. Denote by l the perpendicular bisector of $M'P$, which is necessarily also the perpendicular bisector of $C'B'$.

$M'P$ is a chord in $\bigcirc PSM$, so the centre of $\bigcirc PSM$ is on l. $C'B'$ is a chord in $\bigcirc A'B'C'$, so the centre of the Euler circle is on l. Finally, note that $\triangle MB'C'$ is isosceles, since M is the midpoint of the hypotenuse of $\triangle BB'C$, implying $MB' = MC$, and also the midpoint of the hypotenuse of $\triangle CC'B$, implying $MC' = MC$. Hence M is also on the perpendicular bisector l of $B'C'$. This concludes the argument that $\bigcirc PSM$ and the Euler circle are tangent at M, and applying the inversion $\mathbf{I}$ gives us that $\bigcirc PSM$ and $\bigcirc ABC$ are tangent at M'.

4267. *Proposed by Leonard Giugiuc.*

Let a, b, c and d be real numbers such that $0 < a, b, c \leq 1$ and $abcd = 1$. Prove that

$$5(a+b+c+d) + \frac{4}{abc+abd+acd+bcd} \geq 21.$$

There were 7 correct solutions and two incorrect submissions, as well as one that made use of Maple. *Some of the solutions were quite complicated. We present the solution by Kee-Wai Lau and Angel Plaza, done independently.*

We first note that $d = 1/abc$ and

$$\begin{aligned}
&(a+b+c+d) - (abc+abd+acd+bcd) \\
&= d(a^2bc + ab^2c + abc^2 + 1 - a^2b^2c^2 - ab - ac - bc) \\
&= d(1-ab)(1-ac)(1-bc) \geq 0,
\end{aligned}$$

so that

$$\frac{a+b+c+d}{abc+abd+acd+bcd} \geq 1.$$

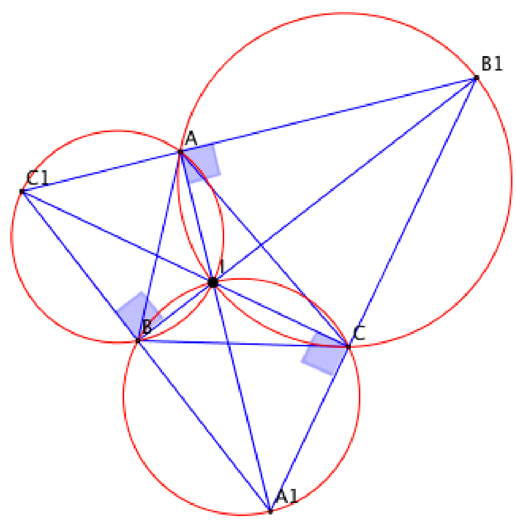
Therefore, applying the arithmetic-geometric means inequality twice, we find that

$$\begin{aligned} &5(a+b+c+d)+\frac{4}{abc+abd+acd+bcd} \\ &= 19\left(\frac{a+b+c+d}{4}\right)+\left[\frac{a+b+c+d}{4}+\frac{4}{abc+abd+acd+bcd}\right] \\ &\geq 19(abcd)^{1/4}+2\left[\frac{a+b+c+d}{abc+abd+acd+bcd}\right]^{1/2} \geq 19+2=21. \end{aligned}$$

4268. *Proposed by Mihaela Berindeanu.*

Let I be the incenter of the acute triangle ABC, and let the triangle's internal angle bisectors intersect the circles IBC, ICA, and IAB again at A_1, B_1, and C_1, respectively. Show that $\overrightarrow{IA_1}+\overrightarrow{IB_1}+\overrightarrow{IC_1}=\overrightarrow{0}$ if and only if $\triangle\, ABC$ is equilateral.

We received 14 submissions, all of which were correct; we feature a composite of similar solutions by Ivko Dimitrić and Titu Zvonaru.



The exterior angle at I of $\triangle AIC$ satisfies $\angle A_1IC = \frac{A+C}{2}$. Since I, B, A_1, C are concyclic (in that order), we have $\angle CA_1I = \angle CBI = B/2$. Hence,

$$\angle ICA_1 = 180^\circ - \frac{A+C}{2} - \frac{B}{2} = 90^\circ$$

and in the same manner, $\angle B_1CI = 90^\circ$, so that $\angle B_1CA_1 = 180^\circ$, which means that the points A_1, C, and B_1 are collinear with $C_1C \perp A_1B_1$. Analogously, B_1, A, C_1 are collinear with $A_1A \perp B_1C_1$, and C_1, B, A_1 are collinear with $B_1B \perp C_1A_1$. Hence, A_1A, B_1B and C_1C are the altitudes of the triangle $A_1B_1C_1$ so that the incenter I of the given triangle ABC is the orthocenter of $\triangle A_1B_1C_1$.

Since for any point I in the plane of triangle $A_1B_1C_1$ one has

$$\overrightarrow{IA_1} + \overrightarrow{IB_1} + \overrightarrow{IC_1} = 3\overrightarrow{IG_1},$$

where G_1 is the centroid of $\triangle A_1B_1C_1$, this sum will be zero if and only if I is the centroid of the said triangle. But, the centroid and incenter of a triangle coincide if and only if the triangle is equilateral. The problem has therefore been reduced to proving that ΔABC is equilateral if and only if $\Delta A_1B_1C_1$ is equilateral. We have ΔABC is equilateral if and only if $\angle AIB = \angle BIC = \angle CIA = 120°$, if and only if $\angle BC_1A = \angle CA_1B = \angle AB_1C = 60°$, if and only if $\Delta A_1B_1C_1$ is equilateral.

Editor's comments.

(1) Only Leonard Giugiuc observed explicitly that there is no need to require that ΔABC be acute (as our featured solution shows).

(2) It is a standard result that the vertices of $\Delta A_1B_1C_1$ are the excenters of ΔABC (see, for example, Chapter X of Roger A. Johnson, *Advanced Euclidean Geometry*), and many of the submissions made use of well-known properties of this pair of triangles to shorten their arguments.

(3) Anna Valkova Tomova used an argument much like our featured solution to extend the result to

> *ΔABC is isosceles with apex at A if and only if there exists a nonzero real number λ for which $\lambda\overrightarrow{IA_1} + \overrightarrow{IB_1} + \overrightarrow{IC_1} = 0$.*

4269. *Proposed by Hung Nguyen Viet.*

Let $x_1, x_2, \ldots, x_n$ be real numbers such that

$$\sin x_1 \cos x_2 + \sin x_2 \cos x_3 + \cdots + \sin x_n \cos x_1 = \frac{n}{2}.$$

Prove that
$$\cos 2x_1 + \cos 2x_2 + \cdots + \cos 2x_n = 0.$$

There were 15 correct solutions submitted, 9 of which had essentially the argument given below. The remainder relied on an inequality forced to equality by the same upper and lower bounds.

With $x_{n+1} = x_1$, we have that

$$\begin{aligned}\sum_{k=1}^{n}(\sin x_k - \cos x_{k+1})^2 &= \sum_{k=1}^{n}(\sin^2 x_k + \cos^2 x_{k+1}) - 2\sum_{k=1}^{n}\sin x_k \cos x_{k+1} \\ &= \sum_{k=1}^{n}(\sin^2 x_k + \cos^2 x_k) - 2(n/2) = n - n = 0,\end{aligned}$$

so that $\sin x_k = \cos x_{k+1}$ for each k. Therefore,

$$\sum_{k=1}^{n}\cos 2x_k = \sum_{k=1}^{n}(\cos^2 x_k - \sin^2 x_k) = \sum_{k=1}^{n}(\cos^2 x_{k+1} - \sin^2 x_k) = 0.$$

4270. *Proposed by Leonard Giugiuc.*

Let k and t be real numbers with $k \in (0,1)$ and $t \in [\frac{\pi}{4}, \frac{\pi}{2}]$. Prove that

$$\int_0^t \frac{\cos x}{x^k} dx \geq \int_0^t \frac{\sin x}{x^k} dx.$$

We received 10 solutions and will feature just one of them here, by Michel Bataille.

Let $t \in [\frac{\pi}{4}, \frac{\pi}{2}]$. For $x \in (0,t]$, we have

$$0 \leq \frac{\cos x}{x^k} \leq \frac{1}{x^k}$$

and $\int_0^t \frac{1}{x^k}\, dx$ exists, hence the integral $\int_0^t \frac{\cos x}{x^k}\, dx$ exists. The integral $\int_0^t \frac{\sin x}{x^k}\, dx$ also exists since

$$\lim_{x\to 0^+} \frac{\sin x}{x^k} = \lim_{x\to 0^+} x^{1-k} \cdot \frac{\sin x}{x} = 0 \cdot 1 = 0.$$

Now, let

$$I = \int_0^{\pi/4} \frac{\cos x - \sin x}{x^k}\, dx, \quad F(t) = \int_{\pi/4}^t \frac{\cos x - \sin x}{x^k}\, dx, \quad G(t) = I + F(t).$$

We are required to prove that $G(t) \geq 0$.

Since

$$x \mapsto \frac{\cos x - \sin x}{x^k}$$

is continuous on $[\frac{\pi}{4}, \frac{\pi}{2}]$, the function F is differentiable on this interval and so is the function G with $G'(t) = F'(t) = \frac{\cos t - \sin t}{t^k}$. For $t \in (\frac{\pi}{4}, \frac{\pi}{2}]$, we have $\cos t < \sin t$, hence $G'(t) < 0$ and therefore G is decreasing on $[\frac{\pi}{4}, \frac{\pi}{2}]$. As a result, it is sufficient to show that $G(\pi/2) \geq 0$. To this aim, we consider

$$\frac{\sqrt{2}}{2} \cdot G(\pi/2) = \int_0^{\pi/4} \frac{\sin(\frac{\pi}{4} - x)}{x^k}\, dx + \int_{\pi/4}^{\pi/2} \frac{\sin(\frac{\pi}{4} - x)}{x^k}\, dx.$$

The substitutions $x = \frac{\pi}{4} - u$ in the first integral and $x = \frac{\pi}{4} + u$ in the second one lead to

$$\frac{\sqrt{2}}{2} \cdot G(\pi/2) = \int_0^{\pi/4} (\sin u) \left(\frac{1}{(\frac{\pi}{4} - u)^k} - \frac{1}{(\frac{\pi}{4} + u)^k} \right) du.$$

But for $u \in (0, \frac{\pi}{4})$, we have

$$(\frac{\pi}{4} + u)^k \geq (\frac{\pi}{4} - u)^k > 0$$

and $\sin u > 0$, hence

$$(\sin u) \left(\frac{1}{(\frac{\pi}{4} - u)^k} - \frac{1}{(\frac{\pi}{4} + u)^k} \right) \geq 0$$

and so $\frac{\sqrt{2}}{2} \cdot G(\pi/2) \geq 0$ and we are done.