SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2017: 43(1), p. 24–27.

4201. *Proposed by Florin Stanescu.*

Let M be a point in the interior of a regular polygon $A_1A_2\dots A_n$ inscribed in the unit circle centered at O, and let A_kB_k be the chord from the vertex A_k through M. Prove that

$$\frac{A_1B_1^2 + A_2B_2^2 + \cdots + A_nB_n^2}{n} \geq \frac{4}{1+OM^2}.$$

All three submissions were correct; we feature the solution by Michele Bataille.

Let $p = 1 - OM^2$, the negation of the power of M with respect to the unit circle. Since $MA_k \cdot MB_k = p$ for $k = 1, 2, \dots, n$, we have

$$A_kB_k^2 = (A_kM + MB_k)^2 = A_kM^2 + \frac{p^2}{A_kM^2} + 2p,$$

and so

$$\sum_{k=1}^{n} A_kB_k^2 = \sum_{k=1}^{n} A_kM^2 + p^2 \sum_{k=1}^{n} \frac{1}{A_kM^2} + 2np. \qquad (1)$$

From Leibniz's relation (see Focus On...No 16, Vol. 41(3), p. 110-113), we deduce

$$\sum_{k=1}^{n} A_kM^2 = nOM^2 + \sum_{k=1}^{n} OA_k^2 = n(1+OM^2), \qquad (2)$$

and from the Cauchy-Schwarz inequality applied to the vectors $\left(\frac{1}{A_1M}, \dots, \frac{1}{A_nM}\right)$ and $(A_1M, \dots, A_nM)$,

$$\sum_{k=1}^{n} \frac{1}{A_kM^2} \geq \frac{n^2}{\sum_{k=1}^{n} A_kM^2}. \qquad (3)$$

(Alternatively, (3) compares the harmonic and arithmetic means: $\frac{1}{HM} \geq \frac{1}{AM}$.) Inserting (2) and (3) into (1) yields

$$\sum_{k=1}^{n} A_kB_k^2 \geq n(1+OM^2) + \frac{n^2(1-OM^2)^2}{n(1+OM^2)} + 2n(1-OM^2);$$

that is,

$$\sum_{k=1}^{n} A_kB_k^2 \geq \frac{4n}{1+OM^2},$$

and the required inequality follows.

Editor's comment. Formula (2) can also be found in "Sums of Squares of Distances" by Tom M. Apostol and Mamikon A. Mnatsakanian [*Math Horizons*, **9**:2 (November 2001)]. It is formula (2) in the proof of their Theorem 1. For the special case of a triangle, it is Theorem 275 on page 174 of Roger A. Johnson's *Advanced Euclidean Geometry*.

4202. *Proposed by Roy Barbara.*

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(a^2 + b^2) = f(a)^2 + f(b)^2$$

for all $a, b \in \mathbb{N}$.

We received 8 correct solutions and will feature just one of them here, by Steven Chow.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. By replacing a and b with 0 in the given equation, $f(0) = 2f(0)^2$, so since $f(0) \in \mathbb{N}$, $f(0) = 0$.

By replacing b with 0 in the given equation, $f\left(a^2\right) = f(a)^2 + f(0)^2 = f(a)^2$. By replacing a with 1, $f(1) = f(1)^2 \implies f(1) \in \{0, 1\}$.

Using mathematical induction, it shall be proved that for all $x \in \mathbb{N}$, $f(x) = xf(1)$.

By replacing a and b with 1 in the given equation, $f(2) = f(1)^2 + f(1)^2 = 2f(1)$. By replacing a with 2 and b with 0, $f(4) = f(2)^2 + f(0)^2 = 4f(1)$. By replacing a with 2 and b with 1, $f(5) = f(2)^2 + f(1)^2 = 5f(1)$. By replacing a with 4 and b with 3, therefore $f(5)^2 = f\left(5^2\right) = f(4)^2 + f(3)^2$, so $f(3) = 3f(1)$. By replacing a and b with 2, $f(8) = f(2)^2 + f(2)^2 = 8f(1)$. By replacing a with 3 and b with 1, $f(10) = f(3)^2 + f(1)^2 = 10f(1)$. By replacing a with 8 and b with 6, therefore $f(10)^2 = f\left(10^2\right) = f(8)^2 + f(6)^2$, so $f(6) = 6f(1)$.

Therefore for all integers $0 \le x \le 6$, $f(x) = xf(1)$. Assume that for some integer $k \ge 6$, for all integers $0 \le x \le k$, $f(x) = xf(1)$.

If $k \equiv 0 \pmod 2$, then

$$\begin{aligned}
&(k+1)^2 + \left(\frac{k}{2} - 2\right)^2 = (k-1)^2 + \left(\frac{k}{2} + 2\right)^2 \\
\implies\ & f\left((k+1)^2 + \left(\frac{k}{2} - 2\right)^2\right) = f\left((k-1)^2 + \left(\frac{k}{2} + 2\right)^2\right) \\
\implies\ & f(k+1)^2 + f\left(\frac{k}{2} - 2\right)^2 = f(k-1)^2 + f\left(\frac{k}{2} + 2\right)^2,
\end{aligned}$$

and $k \ge 6 \implies k \ge \frac{k}{2} - 2, k-1, \frac{k}{2} + 2$, so from the induction hypothesis, therefore $f(k+1) = (k+1)f(1)$.

If $k \equiv 1 \pmod 2$, then

$$\begin{aligned}
&(k+1)^2 + \left(\frac{k-1}{2} - 4\right)^2 = (k-3)^2 + \left(\frac{k-1}{2} + 4\right)^2 \\
\implies\ & f\left((k+1)^2 + \left(\frac{k-1}{2} - 4\right)^2\right) = f\left((k-3)^2 + \left(\frac{k-1}{2} + 4\right)^2\right) \\
\implies\ & f(k+1)^2 + f\left(\frac{k-1}{2} - 4\right)^2 = f(k-3)^2 + f\left(\frac{k-1}{2} + 4\right)^2,
\end{aligned}$$

and $k \geq 7 \implies k \geq \frac{k-1}{2} - 4, k-3, \frac{k-1}{2} + 4$, so from the induction hypothesis, therefore $f(k+1) = (k+1) f(1)$.

Remark: It is easy to come up with those equations using $a^2 + b^2 = c^2 + d^2 \iff (a+c)(a-c) = (d+b)(d-b)$.

Therefore for all $x \in \mathbb{N}$, $f(x) = xf(1)$, so since $f(1) \in \{0, 1\}$, either $f(x) = 0$ for all x, which satisfies the given conditions, or $f(x) = x$ for all x, which satisfies the given conditions.

Therefore all such f are either $f(x) = 0$ for all $x \in \mathbb{N}$, or $f(x) = x$ for all $x \in \mathbb{N}$.

4203. *Proposed by Michel Bataille.*

The incircle of a triangle ABC has centre I, radius r and intersects the line segments AI, BI, CI at A', B', C', respectively. Prove that

(a) $AA' \cdot BB' \cdot CC' \leq \frac{\sqrt{3}}{18}(AB + BC + CA)r^2$;

(b) $A'B' \cdot B'C' \cdot C'A' \leq 3\sqrt{3}\, r^3$.

We received 4 correct solutions to part (a) and 5 to part (b). We present the solution by Leonard Giugiuc.

(a) We have

$$AA' = AI - r = \frac{r}{\sin \frac{A}{2}} - r = \frac{r\left(1 - \sin \frac{A}{2}\right)}{\sin \frac{A}{2}}.$$

Analogously, $BB' = \frac{r\left(1 - \sin \frac{B}{2}\right)}{\sin \frac{B}{2}}$ and $\frac{r\left(1 - \sin \frac{C}{2}\right)}{\sin \frac{C}{2}}$. Hence

$$AA' \cdot BB' \cdot CC' = \frac{r^3\left(1 - \sin \frac{A}{2}\right)\left(1 - \sin \frac{B}{2}\right)\left(1 - \sin \frac{C}{2}\right)}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}.$$

On the other hand,

$$AB + BC + CA = 8R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

Hence the required inequality is equivalent to

$$\frac{3\sqrt{3}r^3\left(1-\sin\frac{A}{2}\right)\left(1-\sin\frac{B}{2}\right)\left(1-\sin\frac{C}{2}\right)}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \le 4Rr^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$
$$\frac{r^3\left(1-\sin\frac{A}{2}\right)\left(1-\sin\frac{B}{2}\right)\left(1-\sin\frac{C}{2}\right)}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \le 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

But $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, so that the last inequality becomes

$$3\sqrt{3}\left(1-\sin\frac{A}{2}\right)\left(1-\sin\frac{B}{2}\right)\left(1-\sin\frac{C}{2}\right) \le \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}. \qquad (1)$$

Set $X = \frac{B+C}{2}$, $Y = \frac{C+A}{2}$, and $Z = \frac{A+B}{2}$. Then X, Y, Z are acute and the sum $X+Y+Z = \pi$. Moreover, (1) becomes

$$3\sqrt{3}(1-\cos X)(1-\cos Y)(1-\cos Z) \le \sin X\sin Y\sin Z$$
$$3\sqrt{3}\sin^2\frac{X}{2}\sin^2\frac{Y}{2}\sin^2\frac{Z}{2} \le \sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}\cos\frac{X}{2}\cos\frac{Y}{2}\cos\frac{Z}{2}$$
$$\cot\frac{X}{2}\cot\frac{Y}{2}\cot\frac{Z}{2} \ge 3\sqrt{3}.$$

But since $\frac{X+Y+Z}{2} = \frac{\pi}{2}$,

$$\cot\frac{X}{2}\cot\frac{Y}{2}\cot\frac{Z}{2} = \cot\frac{X}{2} + \cot\frac{Y}{2} + \cot\frac{Z}{2},$$

and since the cotangent function is convex on $(0, \frac{\pi}{2})$, Jensen's inequality gives

$$\cot\frac{X}{2} + \cot\frac{Y}{2} + \cot\frac{Z}{2} \ge 3\sqrt{3}.$$

(b) The triangle $A'B'C'$ is inscribed in a circle of radius r, and we need to prove that $A'B'\cdot B'C'\cdot A'C' \le 3\sqrt{3}r^3$ or equivalently $4[A'B'C'] \le 3\sqrt{3}r^2$, which is a well-known inequality.

4204. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Let ABC be a triangle with $AB \ne AC$ and let I be the incenter of ABC. Suppose the lines AI, BI and CI intersect the sides BC, CA and AB in D, E and F, respectively. If $DE = DF$ and $\angle ABC = 2\angle ACB$, find $\angle ACB$.

We received 9 submissions, all of which were correct; we present the solution by Madhav Modak, with the final paragraph modified by the editor.

We shall see that the required angle is $\angle ACB = \frac{\pi}{7}$. Let a, b, c denote the lengths of the sides BC, CA, AB respectively. We may assume that $a \ne c$, otherwise we would have $\angle A = \angle C = 45°$ and $\angle B = 90°$, which would imply $DE < DF$. Since $\angle B = 2\angle C$, the sine law gives $c\sin B = b\sin C$, or $2c\sin C\cos C = b\sin C$, or

$$b = 2c\cos C. \qquad (1)$$

Hence $b = 2c(a^2 + b^2 - c^2)/2ab$ or $(a - c)b^2 = c(a^2 - c^2)$. Therefore, since $a \neq c$, we get $b^2 = c(a + c)$ or

$$a = \frac{b^2 - c^2}{c}. \tag{2}$$

Next, applying the cosine law to the triangles ADE and ADF, the condition $DE = DF$ gives

$$\begin{aligned}
&AD^2 + AE^2 - 2AD \cdot AE \cos(A/2) = AD^2 + AF^2 - 2AD \cdot AF \cos(A/2) \iff \\
&AE^2 - AF^2 = 2AD(AE - AF) \cos(A/2) \iff \\
&AE + AF = 2AD \cos(A/2),
\end{aligned}$$

using $AE \neq AF$ (because $AE = AF$ would imply $AB = AC$). Squaring, we have

$$\begin{aligned}
(AE + AF)^2 &= 2AD^2(1 + \cos A) \iff \\
(AE + AF)^2 &= 2AD^2 \left[\frac{(b + c)^2 - a^2}{2bc}\right].
\end{aligned} \tag{3}$$

Using properties of angle bisectors gives us

$$AE = \frac{bc}{c + a}, \quad AF = \frac{bc}{a + b}, \quad AD^2 = bc\left[1 - \frac{a^2}{(b + c)^2}\right].$$

Substituting in (3) we get

$$\begin{aligned}
b^2c^2 \cdot \frac{(2a + b + c)^2}{(a + b)^2(a + c)^2} &= 2bc\left[1 - \frac{a^2}{(b + c)^2}\right] \cdot \left[\frac{(b + c)^2 - a^2}{2bc}\right] \iff \\
b^2c^2 \cdot \frac{(2a + b + c)^2}{(a + b)^2(a + c)^2} &= \left[\frac{(b + c)^2 - a^2}{b + c}\right]^2.
\end{aligned}$$

Taking positive square roots, we get

$$\begin{aligned}
&bc(2a + b + c)(b + c) = (a + b)(a + c)[(b + c)^2 - a^2] \iff \\
&a^3 + (b + c)a^2 - (b^2 + c^2 + bc)a - (b + c)(b^2 + c^2) = 0.
\end{aligned}$$

Using (2) to eliminate a, we get

$$\begin{aligned}
&\frac{b(b + c)^2}{c^3}\left(b^3 - b^2c - 2bc^2 + c^3\right) = 0 \iff \\
&x^3 - x^2 - 2x + 1 = 0, \quad x = b/c = 2\cos C.
\end{aligned} \tag{4}$$

We have seen this equation before in *Crux*; its zeros are $2\cos\frac{\pi}{7}, -2\cos\frac{2\pi}{7}$, and $-2\cos\frac{4\pi}{7}$. It is $P(x) = 0$ where $P(x)$ is essentially the polynomial on page 57 of Michel Bataille's recent article, "About the Side and Diagonals of the Regular Heptagon" [2017 : 55-60]. We conclude that ABC is what has been called the *heptagonal triangle* — the scalene triangle that is formed by three vertices of a regular heptagon. Its angles are $\angle C = \frac{\pi}{7}, \angle B = \frac{2\pi}{7}$, and $\angle A = \frac{4\pi}{7}$.

Editor's comments. Zvonaru remarked that the converse of our problem is Result 14 on page 17 of Leon Bankoff and Jack Garfunkel, "The Heptagonal Triangle", *Mathematics Magazine*, **46**:1 (Jan.-Feb. 1973) 7-19: *The triangle formed by joining the feet of the internal angle bisectors of the heptagonal triangle is isosceles.*

Triangle with $\angle B = 2\angle C$ appeared many times before on the pages of ***Crux***; see J. Chris Fisher's "Recurring Crux Configurations 7: Triangles whose angles satisfy $B = 2C$" [2012: 238-240]. They are characterized by formula (2) above.

4205. *Proposed by Daniel Sitaru.*

Prove that for $0 < a < c < b, a, b, c \in \mathbb{R}$, we have

$$\frac{1}{c\sqrt{ab}}\int_a^b x \arctan x dx > \frac{(c-a)\arctan\sqrt{ac}}{\sqrt{bc}} + \frac{(b-c)\arctan\sqrt{bc}}{\sqrt{ac}}.$$

Ten correct solutions were received. They all followed the same strategy, some depending on the Hermite-Hadamard inequality. Our solution is based on that of Paul Bracken.

Let $f(x) = x \arctan x$ for $x > 0$. Since $f(0) = f'(0) = 0$, $f'(x) = \arctan x + x(1 + x^2)^{-1}$ and $f''(x) = 2(1 + x^2)^{-2}$, then f is positive, strictly increasing and strictly convex. By the Mean Value Theorem, we have that

$$f(p) + f'(p)(x - p) < f(x)$$

for distinct positive x and p. Hence

$$\begin{aligned}(c-a)f(\sqrt{ac}) &< (c-a)f(\sqrt{ac}) + \frac{1}{2}f'(\sqrt{ac})(c-a)(\sqrt{c}-\sqrt{a})^2 \\ &= (c-a)f(\sqrt{ac}) + f'(\sqrt{ac})\int_a^c (x - \sqrt{ac})\,dx \\ &< \int_a^c f(x)\,dx,\end{aligned}$$

and

$$(b-c)f(\sqrt{bc}) < \int_c^b f(x)\,dx.$$

Therefore

$$(c-a)\sqrt{ac}\arctan\sqrt{ac} + (b-c)\sqrt{bc}\arctan\sqrt{bc} < \int_a^b x\arctan x\,dx.$$

Dividing by $(\sqrt{ac})(\sqrt{bc})$ yields the desired inequality.

4206. *Proposed by Gheorghe Alexe and George-Florin Serban.*

Find positive integers p and q that are relatively prime to each other such that $p + p^2 = q + q^2 + 3q^3$.

We received 19 complete solutions. We present the one by Prithwijit De.

We observe that $p+p^2$ is even for any positive integer p. Therefore in any solution q must be even. By rewriting the given equation as

$$p(1 + p) = q(1 + q + 3q^2)$$

we obtain $p|(1 + q + 3q^2)$ and $q|(p + 1)$. We may also rewrite the equation as

$$(p - q)(p + q + 1) = 3q^3$$

which implies $p > q$. Since $\gcd(p - q, q) = \gcd(p, q) = 1$, we can conclude that $q^3|(p + q + 1)$ and therefore

$$q^3 - q - 1 \leq p \leq 1 + q + 3q^2,$$

which leads to

$$q^3 - 3q^2 - 2q - 2 \leq 0.$$

Thus $q \leq 3$ and since q is positive and even, $q = 2$. We obtain $(p, q) = (5, 2)$ as the only solution.

4207. *Proposed by Mihaela Berindeanu.*

Let x, y and z be real numbers such that $x + y + z = 3$. Show that

$$\frac{1}{1 + 2^{4-3x}} + \frac{1}{1 + 2^{4-3y}} + \frac{1}{1 + 2^{4-3z}} \geq 1.$$

We received 18 solutions. We present 2 solutions.

Solution 1, by AN-anduud Problem Solving Group.

We have $2^{4-3x} \cdot 2^{4-3y} \cdot 2^{4-3z} = 8$, hence there exist a, b, c positive real numbers satisfying the following equalities:

$$2^{4-3x} = \frac{2ab}{c^2}, \qquad 2^{4-3y} = \frac{2bc}{a^2}, \qquad 2^{4-3z} = \frac{2ca}{b^2}.$$

The given inequality is equivalent to

$$\begin{aligned} & \frac{1}{1 + \frac{2ab}{c^2}} + \frac{1}{1 + \frac{2bc}{a^2}} + \frac{1}{1 + \frac{2ca}{b^2}} \geq 1 \\ \iff & \frac{c^2}{c^2 + 2ab} + \frac{a^2}{a^2 + 2bc} + \frac{b^2}{b^2 + 2ca} \geq 1 \qquad (1) \end{aligned}$$

Using Cauchy-Schwarz inequality, we get

$$\frac{c^2}{c^2+2ab}+\frac{a^2}{a^2+2bc}+\frac{b^2}{b^2+2ca}\geq\frac{(a+b+c)^2}{(c^2+2ab)+(a^2+2bc)+(b^2+2ca)}=1.$$

Thus inequality (1) is proved. Equality holds if and only if $x=y=z=1$.

Solution 2, by Arkady Alt.

Let $a=2^{4-3x}$, $b=2^{4-3y}$, $c=2^{4-3y}$. Then $a,b,c>0$ and

$$abc=2^{12-3(x+y+z)}=8.$$

The original inequality becomes

$$\sum_{cyc}\frac{1}{1+a}\geq 1 \quad\Longleftrightarrow\quad \sum_{cyc}(1+b)(1+c)\geq(1+a)(1+b)(1+c)$$

The last inequality gives

$$\begin{aligned}3+2(a+b+c)+ab+bc+ca &\geq 1+a+b+c+ab+bc+ca+abc\\ &= 9+a+b+c+ab+bc+ca,\end{aligned}$$

so $a+b+c\geq 6$, which is true because by AM-GM Inequality

$$a+b+c\geq 3\sqrt[3]{abc}=3\sqrt[3]{8}=6.$$

4208. *Proposed by Leonard Giugiuc, Daniel Sitaru and Marian Dinca.*

Let x,y and z be positive real numbers such that $x\leq y\leq z$. Prove that for any real number $k>2$, we have:

$$xy^k+yz^k+zx^k\geq x^2y^{k-1}+y^2z^{k-1}+z^2x^{k-1}.$$

We received 8 solutions. We present the one by Digby Smith.

Since $0<x\leq y\leq z$ and $k>2$, we have $0<x^{k-2}\leq y^{k-2}\leq z^{k-2}$. Thus

$$\begin{aligned}&(xy^k+yz^k+zx^k)-(x^2y^{k-1}+y^2z^{k-1}+z^2x^{k-1})\\ &=xy(y-x)y^{k-2}+yz(z-y)z^{k-2}+zx(x-z)x^{k-2}\\ &\geq xy(y-x)x^{k-2}+yz(z-y)x^{k-2}+zx(x-z)x^{k-2}\\ &=(xy^2-x^2y+yz^2-y^2z+zx^2-xz^2)x^{k-2}\\ &=(z-y)(y-x)(z-x)x^{k-2},\end{aligned}$$

where the last line is clearly non-negative. Hence the desired inequality follows, and clearly equality holds if and only if $x=y=z$.

4209. *Proposed by Nguyen Viet Hung.*

Let m and n be distinct positive integers. Evaluate

$$\lim_{x\to 0} \frac{(1+nx)^m - (1+mx)^n}{\sqrt[m]{1+mx} - \sqrt[n]{1+nx}}.$$

Twenty-one correct solutions were received. The majority followed the approach of Solution 1. Five solvers gave essentially Solution 2, and two had Solution 3.

Solution 1.

Since

$$\begin{aligned}&(1+nx)^m - (1+mx)^n\\ &= [1+mnx+\frac{m(m-1)}{2}n^2x^2+o(x^2)] - [1+mnx+\frac{n(n-1)}{2}m^2x^2+o(x^2)]\\ &= -\frac{mn(n-m)}{2}x^2+o(x^2),\end{aligned}$$

and

$$\begin{aligned}(1+mx)^{1/m} - (1+nx)^{1/n} &= [1+x+\frac{1-m}{2}x^2+o(x^2)] - [1+x+\frac{1-n}{2}x^2+o(x^2)]\\ &= \frac{n-m}{2}x^2+o(x^2)],\end{aligned}$$

the desired limit is $-mn$.

Solution 2.

Let $a \equiv a(x) = \sqrt[n]{1+nx}$ and $b \equiv b(x) = \sqrt[m]{1+mx}$. The expression whose limit is sought is equal to

$$-\left[\frac{a^{mn}-b^{mn}}{a-b}\right] = -[a^{mn-1} + a^{mn-2}b + \cdots + ab^{mn-2} + b^{mn-1}].$$

Since $\lim_{x\to 0} a(x) = \lim_{x\to 0} b(x) = 1$, the desired limit is $-mn$.

Solution 3.

Using l'Hôpital's Rule twice, we find that

$$\begin{aligned}&\lim_{x\to 0} \frac{(1+nx)^m - (1+mx)^n}{(1+mx)^{1/m} - (1+nx)^{1/n}}\\ &\quad= mn \lim_{x\to 0} \frac{(1+nx)^{m-1} - (1+mx)^{n-1}}{(1+mx)^{(1/m)-1} - (1+nx)^{(1/n)-1}}\\ &\quad= mn \lim_{x\to 0} \frac{n(m-1)(1+nx)^{m-2} - m(n-1)(1+mx)^{n-2}}{(1-m)(1+mx)^{(1/m)-2} - (1-n)(1+nx)^{(1/n)-2}}\\ &\quad= mn \frac{n(m-1)-m(n-1)}{(1-m)-(1-n)} = -mn.\end{aligned}$$

4210. *Proposed by Van Khea and Leonard Giugiuc.*

Let ABC be a triangle in which the circumcenter lies on the incircle. Furthermore, let $BC = a, CA = b$ and $AB = c$. For which triangles does the expression $\dfrac{a+b+c}{\sqrt[3]{abc}}$ attain its minimum?

We received 6 submissions, of which 5 were correct and one was incomplete. We present the solution by Arkady Alt.

Let I, O, r, R, and S denote the incenter, circumcenter, inradius, circumradius, and semiperimeter of ΔABC, respectively. It is known (Euler's Theorem) that $OI^2 = R^2 - 2Rr$. By assumption, O lies on the incircle of ΔABC, so $OI = r$. Hence, $R^2 - 2Rr = r^2$ if and only if

$$\left(\frac{R}{r}\right)^2 - 2\left(\frac{R}{r}\right) - 1 = 0 \iff \frac{R}{r} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2} \iff R = \left(\sqrt{2}+1\right) r,$$

since $R \neq (1-\sqrt{2})r$. It is known (Emmerich's Inequality, pg. 251 in *Recent Advances in Geometric Inequalities* by D. S. Mitrinović, J. Pečarić, V. Volenec) that for any non-acute triangle T, we have $R \geq \left(\sqrt{2}+1\right) r$ with equality if and only if T is a right isosceles triangle. Hence we may now assume that ΔABC is non-obtuse. Then $\cos A \cos B \cos C \geq 0$. Since

$$\cos A \cos B \cos C = \frac{S^2 - (2R+r)^2}{4R^2},$$

we have $S \geq 2R + r$ which by $R = (\sqrt{2}+1)r$ implies

$$S \geq 2(\sqrt{2}+1)r + r = (2\sqrt{2}+3)r \quad \text{or} \quad \frac{S}{r} \geq 2\sqrt{2}+3.$$

Therefore,

$$\begin{aligned}\frac{(a+b+c)^3}{abc} = \frac{8S^3}{4RrS} = \frac{2S^2}{Rr} = \frac{2S^2}{(\sqrt{2}+1)r^2} &\geq \frac{2(2\sqrt{2}+3)^2}{\sqrt{2}+1} \\ &= 2(17+12\sqrt{2})(\sqrt{2}-1) = 14+10\sqrt{2}.\end{aligned}$$

Since equality holds in $\frac{S}{r} \geq 2\sqrt{2}+3$ if and only if $\cos A \cos B \cos C = 0$, we conclude that the lower bound $14 + 10\sqrt{2}$ can be attained only for a right angled triangle. Without loss of generality, we may assume that $C = 90°$.

Since $2R = a + b - 2r$, we have

$$a + b = 2(R+r) = 2((\sqrt{2}+1)r + r) = 2\sqrt{2}(\sqrt{2}+1)r. \tag{1}$$

Also,

$$ab = 2Sr = 2(2\sqrt{2}+3)r^2 = 2(\sqrt{2}+1)^2 r^2. \tag{2}$$

From (1) and (2) we obtain $(a-b)^2 = (a+b)^2 - 4ab = 0$. Hence,

$$a = b = \sqrt{2}(\sqrt{2}+1)r \quad \text{and} \quad c = \sqrt{a^2+b^2} = \sqrt{2}a = 2(\sqrt{2}+1)r.$$

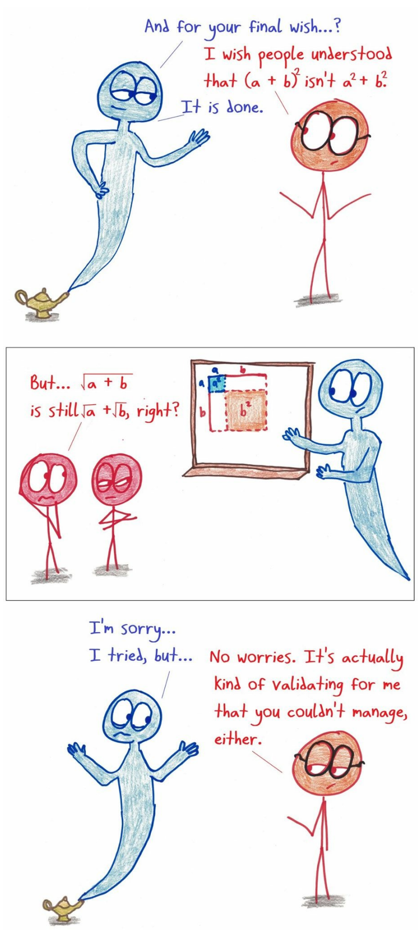
Finally, since

$$(\sqrt[3]{2}+\sqrt[6]{32})^3 = 2+3(2^{2/3})(2^{5/6})+3(2^{1/3})(2^{5/3})+4(2^{1/2})$$
$$= 2+3(2^2)+3(2^{3/2})+4(2^{1/2}) = 14+10\sqrt{2},$$

we have

$$\min\frac{(a+b+c)}{\sqrt[3]{abc}} = \sqrt[3]{14+10\sqrt{2}} = \sqrt[3]{2}+\sqrt[6]{32},$$

attained if and only if ΔABC is a right angled isosceles triangle.



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