Application of Hadamard's Theorems to inequalities

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In this note we will develop a technique based on Hadamard's Theorems for solving some inequalities.

1 Introduction

This method is based on identifying an inequality member as being a possible squared value of a determinant. We will call it *the determinant of a key matrix*. After applying one of Hadamard's theorems, one of the inequality members will be the squared key matrix determinant, and the other one will be the sum of the squared elements on the key matrix columns, or one of the inequality members will be the determinant value and the other one will be the product of the elements on the main diagonal of the key matrix. The trick, of course, is identifying the key matrix.

In 1893, Hadamard published Theorems 1 and 2 in [1]. The consequences of those theorems are extensively analysed in [2]. A recent proof of Theorem 1 can be found in [3]. An innovative approach to proving Theorem 2 can be found in [4].

Theorem 1 *([1, 2, 3]) If $A \in M_n(\mathbb{R})$ then*

$$(\det A)^2 \leqslant \prod_{j=1}^{n} (\sum_{i=1}^{n} a_{ij}^2) \quad . \tag{1}$$

Definition 1 *([4]) A square matrix A is* positive *if $\det A \geq 0$ and $a_{ii} \geq 0$ for $i = 1, \ldots, n$.*

Theorem 2 *([4]) If $A \in M_n(\mathbb{R})$ is positive, then*

$$\det A \leqslant \prod_{i=1}^{n} a_{ii}, \tag{2}$$

with equality if and only if A is a diagonal matrix.

2 Examples

Example 1 *Prove that if a, b, c and d are real numbers, then*

$$(\sin 2a - \sin 2b)^2 \leqslant 4(\sin^2 a + \cos^2 b)(\sin^2 b + \cos^2 a).$$

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Proof. Substituting $\sin x = 2 \sin x \cos x$ on the left, and cancelling the 4's, this inequality is equivalent to:

$$(\sin a \cos a - \sin b \cos b)^2 \leqslant (\sin^2 a + \cos^2 b)(\sin^2 b + \cos^2 b)\ .$$

We recognize the left side as a possible square of a determinant, specifically that of

$$A = \begin{pmatrix} \sin a & \sin b \\ \cos b & \cos a \end{pmatrix} = (a_{ij})_{\substack{1 \leqslant i \leqslant 2 \\ 1 \leqslant j \leqslant 2}}.$$

Now using (1), we get:

$$(\sin a \cos a - \sin b \cos b)^2 \leqslant (\sin^2 a + \cos^2 b)(\sin^2 b + \cos^2 a).$$

□

Example 2 *Prove that if a, b and c are real numbers, then*

$$(3abc - a^3 - b^3 - c^3)^2 \leqslant (a^2 + b^2 + c^2)^3.$$

Proof. As we can see, the expression $3abc - a^3 - b^3 - c^3$ is plausibly the sum of the threefold products each involving some mixture of a, b, c. This structure reminds us of a determinant. The three summands are positive, suggesting each forward extended diagonal has one each of a, b, c. The three cubes are negative, suggesting the back diagonals with repeated values. Let

$$A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}.$$

Then $\det A = 3abc - a^3 - b^3 - c^3$. From Hadamard's first theorem, we have

$$(\det A)^2 \leqslant \prod_{j=1}^{3} (\sum_{i=1}^{3} a_{ij}^2).$$

□

Example 3 *Prove that if $0 < a \leqslant b \leqslant c$, then*

$$(b - a)(c - a)(c - b) < bc^2.$$

Proof. As we can see, the expression $(b - a)(c - a)(c - b)$ is plausibly the value of a Vandermonde determinant

$$V(a, b, c) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

Since A is a positive matrix, so $\det A > 0, b > 0, c^2 > 0$. From Hadamard's second theorem,

$$\det A \leqslant \prod_{i=1}^{n} a_{ii} = bc^2$$

It follows that $\det A \leqslant 1 \cdot b \cdot c^2$ and $(b - a)(c - a)(c - b) < bc^2$. □

3 Proposed problems

Problem 1. Prove that if a, b, c and d are real numbers, then

$$(ad - bc)^2 \leqslant (a^2 + c^2)(b^2 + d^2).$$

Problem 2. Prove that if $a, b, c, d \in (1, \infty)$, then

$$(e^a \ln b - e^b \ln a)^2 \leqslant (e^{2a} + e^{2b})(\ln^2 a + \ln^2 b).$$

Problem 3. Prove that if a, b and c are real numbers, then

$$(2 - a - b - c + abc)^2 \leqslant (a^2 + 2)(b^2 + 2)(c^2 + 2).$$

Problem 4. Prove that if a, b, c and d are positive real numbers, then

$$(abc - ac - bc - ac)^2 \leqslant 4(1 + a^2)(1 + b^2)(1 + c^2).$$

Problem 5. Prove that if n is a positive natural number and $a > 1$, then

$$(n + a - 1)(a - 1)^{n-1} \leqslant a^n.$$

Hint. The key matrices are, in order:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e^a & \ln a \\ e^b & \ln b \end{pmatrix}, \begin{pmatrix} a & a & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & b & 0 \\ 1 & 0 & 0 & c \end{pmatrix}, \begin{pmatrix} a & 1 & 1 & \dots & 1 \\ 1 & a & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & a \end{pmatrix}.$$

References

[1] J. Hadamard, *Résolution d'une question relative aux determinants.* In: Buletin des sciences math (2), 17, 1893, pp. 240-248.

[2] V. Maz'ya and T.O. Shaposnikova, *Jacques Hadamard: "A Universal Mathematician"*, 1998, American Mathematical Society/London Mathematical Society, ISBN 978-0821808412, pp. 383.

[3] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products.* Sixth edition. Academic Press, San Diego, 2000, pp. 1110.

[4] H. Finbarr, *Another Proof of Hadamard's Determinant Inequality.* Irish Math. Soc. Bulletin 59, Ireland, 2007, pp. 61-64.

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