THE OLYMPIAD CORNER

No. 359

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **June 1, 2018**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC361. Let $n \geq 2$ be a positive integer and define k to be the number of primes less than or equal to n. Let A be a subset of $S = \{2, \ldots, n\}$ such that $|A| \leq k$ and no two elements in A divide each other. Show that one can find a set B of cardinality k such that $A \subseteq B \subseteq S$ and no two elements in B divide each other.

OC362. Given a positive prime number p, prove that there exists a positive integer α such that $p|\alpha(\alpha-1)+3$ if and only if there exists a positive integer β such that $p|\beta(\beta-1)+25$.

OC363. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(y)f(x+f(y)) = f(x)f(xy)$$

for all positive real numbers x and y.

OC364. Consider an acute triangle ABC. Suppose $AB < AC$, let I be the incenter, D the foot of perpendicular from I to BC, and suppose that altitude AH meets BI and CI at P and Q, respectively. Let O be the circumcenter of $\triangle IPQ$, extend AO to meet BC at L and suppose that the circumcircle of $\triangle AIL$ meets BC again at N. Prove that $\frac{BD}{CD} = \frac{BN}{CN}$.

OC365. A square $ABCD$ is divided into n^2 equal small (fundamental) squares by drawing lines parallel to its sides. The vertices of the fundamental squares are called vertices of the grid. A rhombus is called *nice* when:

1. it is not a square;
2. its vertices are points of the grid;
3. its diagonals are parallel to the sides of the square $ABCD$.

Find (as a function of n) the number of nice rhombuses (n is a positive integer greater than 2).

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OC361. Soit n un entier ($n \geq 2$) et soit k le nombre de nombres premiers inférieurs ou égaux à n. Soit A un sous-ensemble de $S = \{2, \ldots, n\}$ tel que $|A| \leq k$ et que A ne contienne pas un élément qui est un diviseur d'un autre élément. Démontrer qu'il existe un ensemble B tel que $|B| = k$, $A \subseteq B \subseteq S$ et que B ne contienne pas un élément qui est un diviseur d'un autre élément.

OC362. Soit p un nombre premier. Démontrer qu'il existe un entier strictement positif α tel que $p|\alpha(\alpha-1)+3$ si et seulement si il existe un entier strictement positif β tel que $p|\beta(\beta-1)+25$.

OC363. Déterminer toutes les fonctions $f : \mathbb{R}^+ \to \mathbb{R}^+$ telles que

$$f(y)f(x + f(y)) = f(x)f(xy)$$

pour tous réels x et y strictement positifs.

OC364. On considère un triangle acutangle ABC où $AB < AC$. Soit I le centre du cercle inscrit dans le triangle et D le pied de la perpendiculaire abaissée de I jusqu'à BC. La hauteur AH coupe BI et CI aux points respectifs P et Q. Soit O le centre du cercle circonscrit au triangle IPQ. On prolonge AO jusqu'au point L sur BC. Le cercle circonscrit au triangle AIL coupe BC de nouveau au point N. Démontrer que $\frac{BD}{CD} = \frac{BN}{CN}$.

OC365. On divise un carré $ABCD$ en n^2 petits carrés en traçant des segments parallèles à ses côtés, formant ainsi un quadrillage. Les points d'intersection du quadrillage sont appelés des points de treillis. On dit qu'un losange est *plaisant* lorsque:

1. il n'est pas un carré;
2. ses sommets sont des points de treillis;
3. ses diagonales sont parallèles aux côtés du carré $ABCD$.

Déterminer, en fonction de n, le nombre de losanges plaisants, n étant un entier supérieur à 2.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(9), p. 385 – 386.

OC301. Solve the following Diophantine equation for integers x and y:

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

Originally Day 1 Problem 1 of the 2015 United States of America Mathematical Olympiad.

We received four correct submissions and we present the solution by Prithwijit De.

Since $3|x + y$, we set $x + y = 3u$, where $u \in \mathbb{Z}$. Then $3(x + y)^2 + (x - y)^2 = 4(x^2 + xy + y^2)$ gives

$$\begin{aligned}(x-y)^2 &= 4(x^2+xy+y^2) - 3(x+y)^2 \\ &= 4(u+1)^3 - 27u^2 \\ &= 4u^3 - 15u^2 + 12u + 4 \\ &= (u-2)^2(4u+1).\end{aligned}$$

Hence, $4u+1$ is an odd perfect square. Let $4u+1 = (2k+1)^2$ where $k \in \mathbb{Z}$. Then $u = k(k+1)$. Therefore

$$x + y = 3k(k+1) = 3k^2 + 3k \qquad (1)$$

and

$$\begin{aligned}x - y = \sqrt{(u-2)^2(4u+1)} = \pm(u-2)\sqrt{4u+1} &= \pm(k^2+k-2)(2k+1) \\ &= \pm(2k^3+3k^2-3k-2). \qquad (2)\end{aligned}$$

Solving (1) and (2), we conclude that the solutions are given by the following set S of all unordered pairs:

$$S = \left\{(x,y) = (k^3 + 3k^2 - 1, -k^3 + 3k + 1) | k \in \mathbb{Z}\right\}$$

OC302. Let x, y and z be real numbers where the sum of any two among them is not 1. Show that

$$\frac{(x^2+y)(x+y^2)}{(x+y-1)^2} + \frac{(y^2+z)(y+z^2)}{(y+z-1)^2} + \frac{(z^2+x)(z+x^2)}{(z+x-1)^2} \geq 2(x+y+z) - \frac{3}{4}.$$

Find all triples (x, y, z) of real numbers satisfying the equality case.

Originally Day 1 Problem 2 of the 2015 Turkey Mathematical Olympiad

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We received six correct submissions and we present a composite of the solutions by Mohammed Aassila and Michel Bataille, expanded slightly by the editor.

Let $2a = x + y$ and $b^2 = xy$. Then by straightforward computations we have

$$\begin{aligned}(x^2+y)(x+y^2) &= x^3+y^3+x^2y^2+xy\\ &= (x+y)^3-3xy(x+y)+x^2y^2+xy\\ &= 8a^3-6ab^2+b^4+b^2, \qquad (1)\end{aligned}$$

and

$$\begin{aligned}(x+y-1)^2(x+y-1/4) &= (2a-1)^2(2a-1/4)\\ &= (4a^2-4a+1)(2a-1/4)\\ &= 8a^3-9a^2+3a-1/4. \qquad (2)\end{aligned}$$

From (1) and (2) we obtain

$$\begin{aligned}&(x^2+y)(x+y^2)-(x+y-1)^2(x+y-1/4)\\ &= 9a^2+b^4+b^2-6ab^2-3a+1/4 = (1/4)(6a-2b^2-1)^2 \geq 0,\end{aligned}$$

so

$$\frac{(x^2+y)(x+y^2)}{(x+y-1)^2} \geq x+y-1/4.$$

Hence,

$$\frac{(x^2+y)(x+y^2)}{(x+y-1)^2}+\frac{(y^2+z)(y+z^2)}{(y+z-1)^2}+\frac{(z^2+x)(z+x^2)}{(z+x-1)^2} \geq 2(x+y+z)-\frac{3}{4}.$$

Since

$$6a-2b^2-1=0 \iff 3(x+y)-2xy-1=0 \iff 3(x+y)=2xy+1,$$

we see that equality holds in the given inequality if and only if

$$3(x+y)=2xy+1, \quad 3(y+z)=2yz+1, \quad \text{and} \quad 3(z+x)=2zx+1.$$

Subtracting, we have

$$3(x-z)=2y(x-z) \quad \textit{or} \quad (x-z)(2y-3)=0.$$

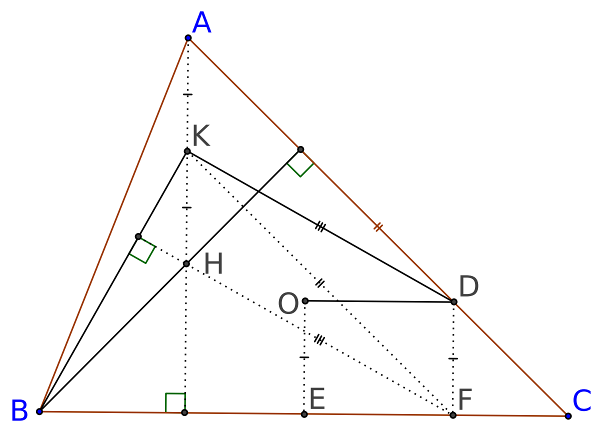
But $2y-3 \neq 0$ since $y = 3/2$ implies $3(x+3/2) = 3x+1$, a contradiction. Hence, $x = z$. Similarly, $y = z$, so $x = y = z$. Then solving $2x^2 - 6x + 1 = 0$ we get $x = \frac{3\pm\sqrt{7}}{2}$. In conclusion, equality holds if and only if

$$x=y=z=\frac{3+\sqrt{7}}{2} \quad \text{or} \quad x=y=z=\frac{3-\sqrt{7}}{2}.$$

OC303. Let ABC be a triangle with orthocenter H and circumcenter O. Let K be the midpoint of AH. Point P lies on AC such that $\angle BKP = 90^\circ$. Prove that $OP \parallel BC$.

Originally Problem 2 of the 2015 Iranian Mathematical Olympiad (Geometry).

We received eight submissions, all of which were correct and complete, although one was completed by a computer, so maybe it should not count. We present a composite of the similar solutions submitted by Daniel Dan and Oliver Geupel.



Define D to be the point where the parallel to the side BC through O meets the line AC, and let E and F denote the feet of the perpendiculars from the points O and D onto the line BC. We are to prove that D coincides with P; that is, we must show that KD is perpendicular to BK. Because $AH = 2OE$ and both lines are perpendicular to BC, it follows that the line segments AK, KH, OE, and DF are parallel and have equal lengths. Hence $AD \parallel KF$ and $HF \parallel KD$. Since $BH \perp AD$, we deduce that $BH \perp KF$. Moreover, we have $KH \perp BF$. Thus, H is the orthocenter of $\triangle BFK$. We obtain $HF \perp BK$ and, therefore, $KD \perp BK$, as desired.

Editor's comment. Most of the other solutions used some form of coordinates. It turned out that almost any approach — Cartesian or triangular coordinates, vectors, or complex numbers — leads to an attractive solution. There is no need for help from *MAPLE*; the use of a computer seems counter to the spirit of Olympiad problem solving.

OC304. Let k be a fixed positive integer. Let $F(n)$ be the smallest positive integer greater than kn such that $F(n) \cdot n$ is a perfect square. Prove that if $F(n) = F(m)$, then $m = n$.

Originally Day 1 Problem 2 of the 2015 Serbian National Mathematical Olympiad.

Two correct solutions were received. The solution below mainly follows that of Kathleen Lewis.

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Suppose, if possible, that $m < n$ and $F(m) = F(n) = u$. Let $m = ra^2$ and $n = sb^2$, where r and s are both squarefree. Since um is square, u must be divisible by r, so that $u = rc^2$ for some integer c. Likewise, $u = sd^2$ for some integer d. Since $(rc)^2 = rsd^2$, rs is square and so $r = s$. Thus, also, $c = d$.

Since $m < n$, then $a < b$ so that $b - a \geq 1$. Since $k \geq 1$, then $b\sqrt{k} - a\sqrt{k} \geq 1$ and there exists a positive integer v for which $a\sqrt{k} < v \leq b\sqrt{k}$. Therefore

$$km = rka^2 < rv^2 \leq rkb^2 = ksb^2 = kn < F(n) = F(m).$$

But this is a contradiction, since $mrv^2 = (rav)^2$ is square so that rv^2 should be not less than $F(m)$.

OC305. Let p be a prime number for which $\frac{p-1}{2}$ is also prime, and let a, b, c be integers not divisible by p. Prove that there are at most $1 + \sqrt{2p}$ positive integers n such that $n < p$ and p divides $a^n + b^n + c^n$.

Originally Problem 5 of the 2015 Canadian Mathematical Olympiad.

No correct solutions were received.