OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(9), p. 385 – 386.

OC301. Solve the following Diophantine equation for integers x and y:

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

Originally Day 1 Problem 1 of the 2015 United States of America Mathematical Olympiad.

We received four correct submissions and we present the solution by Prithwijit De.

Since $3|x + y$, we set $x + y = 3u$, where $u \in \mathbb{Z}$. Then $3(x + y)^2 + (x - y)^2 = 4(x^2 + xy + y^2)$ gives

$$\begin{aligned}(x-y)^2 &= 4(x^2+xy+y^2) - 3(x+y)^2 \\ &= 4(u+1)^3 - 27u^2 \\ &= 4u^3 - 15u^2 + 12u + 4 \\ &= (u-2)^2(4u+1).\end{aligned}$$

Hence, $4u+1$ is an odd perfect square. Let $4u+1 = (2k+1)^2$ where $k \in \mathbb{Z}$. Then $u = k(k + 1)$. Therefore

$$x + y = 3k(k+1) = 3k^2 + 3k \tag{1}$$

and

$$\begin{aligned}x - y = \sqrt{(u-2)^2(4u+1)} = \pm(u-2)\sqrt{4u+1} &= \pm(k^2+k-2)(2k+1) \\ &= \pm(2k^3 + 3k^2 - 3k - 2). \quad (2)\end{aligned}$$

Solving (1) and (2), we conclude that the solutions are given by the following set S of all unordered pairs:

$$S = \left\{(x, y) = (k^3 + 3k^2 - 1, -k^3 + 3k + 1) | k \in \mathbb{Z}\right\}$$

OC302. Let x, y and z be real numbers where the sum of any two among them is not 1. Show that

$$\frac{(x^2+y)(x+y^2)}{(x+y-1)^2} + \frac{(y^2+z)(y+z^2)}{(y+z-1)^2} + \frac{(z^2+x)(z+x^2)}{(z+x-1)^2} \geq 2(x+y+z) - \frac{3}{4}.$$

Find all triples (x, y, z) of real numbers satisfying the equality case.

Originally Day 1 Problem 2 of the 2015 Turkey Mathematical Olympiad

We received six correct submissions and we present a composite of the solutions by Mohammed Aassila and Michel Bataille, expanded slightly by the editor.

Let $2a = x + y$ and $b^2 = xy$. Then by straightforward computations we have

$$\begin{aligned}(x^2 + y)(x + y^2) &= x^3 + y^3 + x^2y^2 + xy \\ &= (x + y)^3 - 3xy(x + y) + x^2y^2 + xy \\ &= 8a^3 - 6ab^2 + b^4 + b^2, \qquad (1)\end{aligned}$$

and

$$\begin{aligned}(x + y - 1)^2(x + y - 1/4) &= (2a - 1)^2(2a - 1/4) \\ &= (4a^2 - 4a + 1)(2a - 1/4) \\ &= 8a^3 - 9a^2 + 3a - 1/4. \qquad (2)\end{aligned}$$

From (1) and (2) we obtain

$$\begin{aligned}&(x^2 + y)(x + y^2) - (x + y - 1)^2(x + y - 1/4) \\ &= 9a^2 + b^4 + b^2 - 6ab^2 - 3a + 1/4 = (1/4)(6a - 2b^2 - 1)^2 \geq 0,\end{aligned}$$

so

$$\frac{(x^2 + y)(x + y^2)}{(x + y - 1)^2} \geq x + y - 1/4.$$

Hence,

$$\frac{(x^2 + y)(x + y^2)}{(x + y - 1)^2} + \frac{(y^2 + z)(y + z^2)}{(y + z - 1)^2} + \frac{(z^2 + x)(z + x^2)}{(z + x - 1)^2} \geq 2(x + y + z) - \frac{3}{4}.$$

Since

$$6a - 2b^2 - 1 = 0 \iff 3(x + y) - 2xy - 1 = 0 \iff 3(x + y) = 2xy + 1,$$

we see that equality holds in the given inequality if and only if

$$3(x + y) = 2xy + 1, \quad 3(y + z) = 2yz + 1, \quad \text{and} \quad 3(z + x) = 2zx + 1.$$

Subtracting, we have

$$3(x - z) = 2y(x - z) \quad \textit{or} \quad (x - z)(2y - 3) = 0.$$

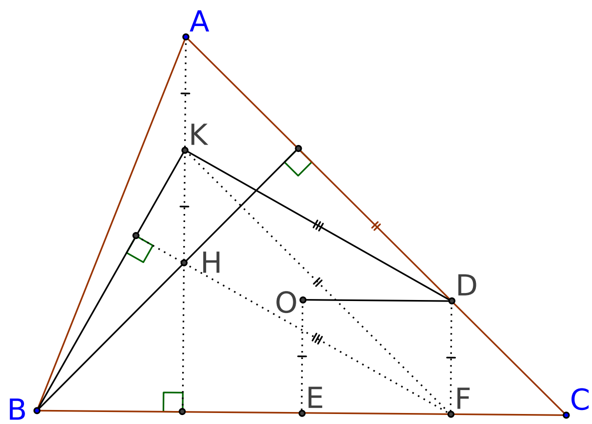
But $2y - 3 \neq 0$ since $y = 3/2$ implies $3(x + 3/2) = 3x + 1$, a contradiction. Hence, $x = z$. Similarly, $y = z$, so $x = y = z$. Then solving $2x^2 - 6x + 1 = 0$ we get $x = \frac{3\pm\sqrt{7}}{2}$. In conclusion, equality holds if and only if

$$x = y = z = \frac{3 + \sqrt{7}}{2} \quad \text{or} \quad x = y = z = \frac{3 - \sqrt{7}}{2}.$$

OC303. Let ABC be a triangle with orthocenter H and circumcenter O. Let K be the midpoint of AH. Point P lies on AC such that $\angle BKP = 90°$. Prove that $OP \parallel BC$.

Originally Problem 2 of the 2015 Iranian Mathematical Olympiad (Geometry).

We received eight submissions, all of which were correct and complete, although one was completed by a computer, so maybe it should not count. We present a composite of the similar solutions submitted by Daniel Dan and Oliver Geupel.



Define D to be the point where the parallel to the side BC through O meets the line AC, and let E and F denote the feet of the perpendiculars from the points O and D onto the line BC. We are to prove that D coincides with P; that is, we must show that KD is perpendicular to BK. Because $AH = 2OE$ and both lines are perpendicular to BC, it follows that the line segments AK, KH, OE, and DF are parallel and have equal lengths. Hence $AD \parallel KF$ and $HF \parallel KD$. Since $BH \perp AD$, we deduce that $BH \perp KF$. Moreover, we have $KH \perp BF$. Thus, H is the orthocenter of $\triangle BFK$. We obtain $HF \perp BK$ and, therefore, $KD \perp BK$, as desired.

Editor's comment. Most of the other solutions used some form of coordinates. It turned out that almost any approach — Cartesian or triangular coordinates, vectors, or complex numbers — leads to an attractive solution. There is no need for help from *MAPLE*; the use of a computer seems counter to the spirit of Olympiad problem solving.

OC304. Let k be a fixed positive integer. Let $F(n)$ be the smallest positive integer greater than kn such that $F(n) \cdot n$ is a perfect square. Prove that if $F(n) = F(m)$, then $m = n$.

Originally Day 1 Problem 2 of the 2015 Serbian National Mathematical Olympiad.

Two correct solutions were received. The solution below mainly follows that of Kathleen Lewis.

Suppose, if possible, that $m < n$ and $F(m) = F(n) = u$. Let $m = ra^2$ and $n = sb^2$, where r and s are both squarefree. Since um is square, u must be divisible by r, so that $u = rc^2$ for some integer c. Likewise, $u = sd^2$ for some integer d. Since $(rc)^2 = rsd^2$, rs is square and so $r = s$. Thus, also, $c = d$.

Since $m < n$, then $a < b$ so that $b - a \geq 1$. Since $k \geq 1$, then $b\sqrt{k} - a\sqrt{k} \geq 1$ and there exists a positive integer v for which $a\sqrt{k} < v \leq b\sqrt{k}$. Therefore

$$km = rka^2 < rv^2 \leq rkb^2 = ksb^2 = kn < F(n) = F(m).$$

But this is a contradiction, since $mrv^2 = (rav)^2$ is square so that rv^2 should be not less than $F(m)$.

OC305. Let p be a prime number for which $\frac{p-1}{2}$ is also prime, and let a, b, c be integers not divisible by p. Prove that there are at most $1 + \sqrt{2p}$ positive integers n such that $n < p$ and p divides $a^n + b^n + c^n$.

Originally Problem 5 of the 2015 Canadian Mathematical Olympiad.

No correct solutions were received.