FOCUS ON...

No. 29

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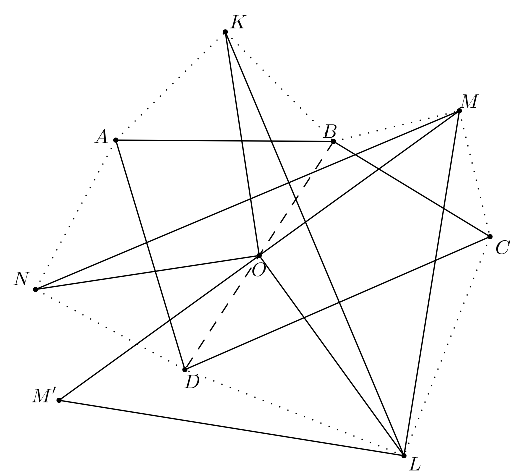
Solutions to Exercises from Focus On... No. 22–26

From Focus On... No. 22

1. *Let $ABCD$ be a convex quadrilateral that is not a parallelogram. On the sides AB, BC, CD, DA, construct isosceles triangles KAB, MBC, LCD, NDA external to $ABCD$ such that the angles at K, L, M, N are right angles. Show that if O is the midpoint of BD, then one of the triangles MON or LOK is a* 90° *rotation of the other around O.*



Without loss of generality, we assume that $ABCD$ is clockwise oriented as on the diagram above. Let $\mathbf{r}_X$ denote the rotation with positive right angle and centre X. The isometry $\mathbf{r}_L \circ \mathbf{r}_M$ is a 180° rotation and

$$\mathbf{r}_L \circ \mathbf{r}_M(B) = \mathbf{r}_L(C) = D.$$

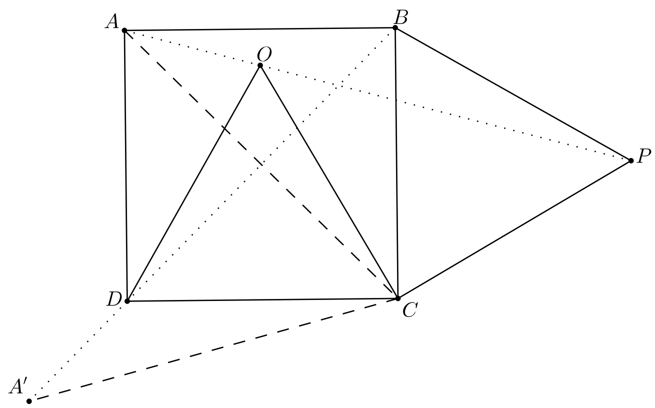
Hence $\mathbf{r}_L \circ \mathbf{r}_M$ is the symmetry $\mathbf{s}_O$ about the mid-point O of BD. Let $M' = \mathbf{s}_O(M)$. Then

$$M' = \mathbf{r}_L \circ \mathbf{r}_M(M) = \mathbf{r}_L(M)$$

so that $\Delta MLM'$ is an isosceles right-angled triangle with right angle at L and $\mathbf{r}_O(L) = M$. Similarly, $\mathbf{r}_O(K) = N$ and therefore the triangle MON is the image of LOK under the 90° rotation $\mathbf{r}_O$.

Note that if $ABCD$ is a parallelogram, then O is also the mid-point of AC and, as above, $\mathbf{r}_O(N) = L$ and $\mathbf{r}_O(M) = K$. It follows that $\mathbf{s}_O(M) = N$ and $\mathbf{s}_O(K) = L$. Then the triangles MON and LOK are degenerate and $LMKN$ is a square with centre O.

2. *Let $ABCD$ be a square and O, P be such that DOC and BCP are equilateral triangles with O inside $ABCD$ and P external to $ABCD$. Show that A, O, P are collinear. (A possible solution follows from the value of $\angle AOB$ found in problem* ***3458*** *[2009 : 326 ; 2010 : 347]; preferably solve the problem with the help of a well-chosen rotation.)*



We introduce the rotation $\mathbf{r}$ with centre C transforming P into B. Let $\mathbf{r}(A) = A'$. Since $\Delta ACA'$ is equilateral, we have $A'C = A'A$, hence A', B, D are collinear (on the perpendicular bisector of AC). As a result, the points

$$A = \mathbf{r}^{-1}(A'), \qquad P = \mathbf{r}^{-1}(B), \qquad O = \mathbf{r}^{-1}(D)$$

are collinear as well.

From Focus On... No. 23

1. *Given that the polynomial $X^3 - 5X + m$ has two roots z_1, z_2 such that $z_1 + z_2 = 2z_1z_2$, find the value of m and all the roots.*

Denoting by z_3 the third root of $X^3 - 5X + m$, Vieta's formulas give

$$z_1 + z_2 + z_3 = 0, \qquad z_1z_2 + z_2z_3 + z_3z_1 = -5, \qquad z_1z_2z_3 = -m.$$

Now, $z_1z_2 - (z_1 + z_2)^2 = -5$ and, if $z_1 + z_2 = 2z_1z_2$ holds, we obtain

$$z_1z_2 - 4(z_1z_2)^2 = -5$$

so that $z_1z_2 = -1$ or $z_1z_2 = \frac{5}{4}$. In the former case,

$$z_3 = -(z_1 + z_2) = -2z_1z_2 = 2 \quad \text{and} \quad m = -z_1z_2z_3 = 2$$

and, similarly, $z_3 = -\frac{5}{2}$ and $m = \frac{25}{8}$ in the latter case.

Conversely, if $m = 2$ the polynomial writes as $X^3 - 5X + 2$ whose roots are

$$z_3 = 2, \quad z_2 = -1 + \sqrt{2}, \quad z_1 = -1 - \sqrt{2}.$$

If $m = \frac{25}{8}$, the roots of $X^3 - 5X + \frac{25}{8}$ are

$$z_3 = -\frac{5}{2}, \quad z_2 = \frac{5+\sqrt{5}}{4}, \quad z_1 = \frac{5-\sqrt{5}}{4}.$$

In both cases, it is readily checked that $z_1 + z_2 = 2z_1z_2$ holds.

2. *Let $Q(x) \in \mathbb{R}[x]$ and $P(x) = a+bx+cx^2+x^3Q(x)$ where a, b, c are real numbers and $ac \neq 0$. Prove that if all the roots of P are real, then $b^2 > 2ac$. (Hint: if n is the degree of P, consider $x^nP(1/x)$.)*

We first note that the conclusion $b^2 > 2ac$ certainly holds if $ac < 0$, so we suppose that $ac > 0$ from now on.

If $Q(x)$ is the zero polynomial, then $P(x) = a+bx+cx^2$ and we must have $b^2 \geq 4ac$, hence $b^2 > 2ac$ holds. Otherwise, let $n > 2$ be the degree of $P(x)$ so that $Q(x)$ is of degree $n-3$, say $Q(x) = q_0 + q_1x + \cdots + q_{n-3}x^{n-3}$. Then, the polynomial

$$U(x) = x^n \cdot P(1/x) = ax^n + bx^{n-1} + cx^{n-2} + q_0x^{n-3} + \cdots + q_{n-2}x + q_{n-3}$$

also has n real roots (the reciprocals of the roots of $P(x)$), say $x_1, x_2, \ldots, x_n$. From Vieta's formulas, we have

$$x_1 + x_2 + \cdots + x_n = -\frac{b}{a}, \qquad \text{and} \qquad \sum_{1\leq i<j\leq n} x_ix_j = \frac{c}{a}.$$

Using the arithmetic-geometric mean inequality, we deduce

$$\frac{2c}{a} \leq \sum_{1\leq i<j\leq n} (x_i^2 + x_j^2) = (n-1)(x_1^2 + x_2^2 + \cdots x_n^2) = (n-1)\left(\frac{b^2}{a^2} - \frac{2c}{a}\right).$$

It follows that $2nac \leq (n-1)b^2$, that is, $b^2 \geq \frac{n}{n-1} \cdot (2ac)$ and $b^2 > 2ac$ holds.

From Focus On... No. 25

1. *Find all complex numbers λ such that the product of two roots of $x^4-2x^2+\lambda x+3$ is 1.*

The polynomial $a(x) = x^4 - 2x^2 + \lambda x + 3$ has two roots whose product is 1 if and only if $a(x)$ is divisible by some polynomial $b(x)$ of the form $x^2 - \mu x + 1$. The long division of $a(x)$ by $b(x)$ gives

$$a(x) = b(x)(x^2 + \mu x + \mu^2 - 3) + (\lambda + \mu^3 - 4\mu)x + 6 - \mu^2.$$

The remainder is the zero polynomial if and only if $\mu^2 = 6$ and $\mu^3 - 4\mu + \lambda = 0$ for some complex number μ. The elimination of μ is immediate and provides the condition $(\lambda + 2\sqrt{6})(\lambda - 2\sqrt{6}) = 0$ on λ. Thus the suitable values of λ are $2\sqrt{6}$ and $-2\sqrt{6}$.

It is easy to obtain the two roots with product 1: they are $\frac{\sqrt{6}+\sqrt{2}}{2}$ and $\frac{\sqrt{6}-\sqrt{2}}{2}$ if $\lambda = -2\sqrt{6}$, and their opposites in the case when $\lambda = 2\sqrt{6}$.

2. *Find real numbers a_k, b_k $(k = 1, 2, \ldots, 2017)$ such that*

$$\frac{3x^5 - 3x^4 - 2x^2 + 2x + 4}{(x^2 + x + 1)^{2017}} = \sum_{k=1}^{2017} \frac{a_k x + b_k}{(x^2 + x + 1)^k}.$$

Let $f(x)$ denote the rational fraction on the left-hand side. The problem is to calculate the decomposition of $f(x)$ into partial fractions in $\mathbb{R}(x)$. Multiplying both sides by $(x^2 + x + 1)^{2017}$, we obtain

$$3x^5 - 3x^4 - 2x^2 + 2x + 4 = \sum_{k=1}^{2017} (a_k x + b_k)(x^2 + x + 1)^{2017-k}, \qquad (1)$$

which is nothing else than the long division of the numerator of $f(x)$ by $x^2 + x + 1$:

$$3x^5 - 3x^4 - 2x^2 + 2x + 4 = (x^2 + x + 1)q_1(x) + a_{2017}x + b_{2017}.$$

Substituting into (1) and dividing by $x^2 + x + 1$, we see that $a_{2016}x + b_{2016}$ is the remainder in the division of $q_1(x)$ by $x^2 + x + 1$:

$$q_1(x) = (x^2 + x + 1)q_2(x) + a_{2016}x + b_{2016}.$$

Here $q_2(x)$ is of degree 1 and so $q_2(x) = a_{2015}x + b_{2015}$ and $a_k = b_k = 0$ if $k \leq 2014$.

An easy calculation successively provides

$$a_{2017}x + b_{2017} = -2x + 3, \quad q_1(x) = 3x^3 - 6x^2 + 3x + 1,$$
$$a_{2016}x + b_{2016} = 9x + 10, \quad q_2(x) = 3x - 9 = a_{2015}x + b_{2015}.$$

Therefore the decomposition of $f(x)$ is

$$f(x) = \frac{-2x + 3}{(x^2 + x + 1)^{2017}} + \frac{9x + 10}{(x^2 + x + 1)^{2016}} + \frac{3x - 9}{(x^2 + x + 1)^{2015}}.$$

From Focus On... No. 26

1. *Let $p(x) \in \mathbb{R}[x]$ with $\deg(p(x)) \geq 2$. Prove that the graph of the function p cannot have more than one centre of symmetry.*

For the purpose of a contradiction, assume that (x_1, y_1) and (x_2, y_2) are two distinct centres of symmetry. Then for any real number x, we have

$$p(x_i + x) + p(x_i - x) = 2y_i \quad (i = 1, 2),$$

from which we readily deduce that $p(x + h) = p(x) + k$ where $h = 2(x_1 - x_2)$ and $k = 2(y_1 - y_2)$. Note that $y_i = p(x_i)$ so that $h \neq 0$. Differentiating, we get $p'(x + h) = p'(x)$ so that whenever the polynomial $p'(x)$ has a complex root z, it has infinitely many roots, namely the numbers $z + nh$, $n = 0, 1, 2, \ldots$ It follows that $p'(x)$ is either the zero polynomial or a nonzero constant polynomial, in contradiction with $\deg(p(x)) \geq 2$.

2. *Let n be a positive integer and let $a(x) \in \mathbb{R}[x]$ with $\deg(a(x)) = n$. Find $a(n+1)$ given that $a(k) = \frac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$,*

a) using the polynomial $(x+1)a(x) - x$;

b) using a Lagrange interpolation polynomial.

a) By hypothesis, the $n+1$ numbers $0, 1, 2, \ldots, n$ are roots of the polynomial

$$b(x) = (x+1)a(x) - x$$

whose degree is $n+1$, hence $b(x) = \rho \cdot x(x-1)(x-2)\cdots(x-n)$ for some nonzero constant ρ.

Since $b(-1) = 1$, the constant ρ satisfies $1 = \rho \cdot (-1)^{n+1}((n+1)!)$ and therefore

$$b(x) = \frac{(-1)^{n+1}}{(n+1)!} \cdot x(x-1)(x-2)\cdots(x-n).$$

We first deduce

$$(n+2)a(n+1) - (n+1) = b(n+1) = (-1)^{n+1}$$

and finally

$$a(n+1) = \frac{n+1+(-1)^{n+1}}{n+2}.$$

b) The polynomial $a(x)$ is the Lagrange interpolation polynomial associated with

$$x_1 = 0, x_2 = 1, \ldots, x_{n+1} = n \quad \text{and} \quad y_1 = 0, y_2 = \frac{1}{2}, \ldots, y_{n+1} = \frac{n}{n+1}.$$

Thus,

$$a(x) = \sum_{k=0}^{n} \frac{k}{k+1} \cdot \frac{x(x-1)\cdots\widehat{(x-k)}\cdots(x-n)}{k! \cdot (-1)^{n-k}(n-k)!},$$

where the factor with the hat is omitted. A short calculation then gives

$$a(n+1) = \sum_{k=1}^{n} (-1)^{n-k} \frac{k}{k+1} \binom{n+1}{k}$$

or, using $k\binom{n+1}{k} = (n+1)\binom{n}{k-1}$,

$$a(n+1) = (-1)^{n+1}(n+1) \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k+1} \binom{n}{k-1} = (-1)^{n+1}(n+1)f(1),$$

where

$$f(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}x^{k+1}}{k+1} \binom{n}{k-1}.$$

We readily obtain the derivative $f'(x) = x\left((1-x)^n - (-1)^n x^n\right)$ so that

$$\begin{aligned} f(1) &= \int_0^1 x(1-x)^n\,dx + \frac{(-1)^{n+1}}{n+2} \\ &= \int_0^1 (1-u)u^n\,du + \frac{(-1)^{n+1}}{n+2} \\ &= \frac{1}{(n+1)(n+2)} + \frac{(-1)^{n+1}}{n+2}. \end{aligned}$$

As a result, $a(n+1) = \dfrac{n+1+(-1)^{n+1}}{n+2}$, in accordance with part (a).

Math Quotes

There's a touch of the priesthood in the academic world, a sense that a scholar should not be distracted by the mundane tasks of day-to-day living. I used to have great stretches of time to work. Now I have research thoughts while making peanut butter and jelly sandwiches. Sure it's impossible to write down ideas while reading *Curious George* to a two-year-old. On the other hand, as my husband was leaving graduate school for his first job, his thesis advisor told him, "You may wonder how a professor gets any research done when one has to teach, advise students, serve on committees, referee papers, write letters of recommendation, interview prospective faculty. Well, I take long showers."

Susan Landau, "In Her Own Words: Six Mathematicians Comment on Their Lives and Careers." Notices of the AMS, V. 38, no. 7 (September 1991).