THE CONTEST CORNER

No. 61

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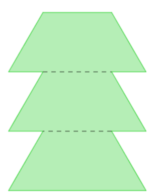
The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **June 1, 2018**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC301. All natural numbers are coloured using 100 different colours. Prove that you can find several (no less than 2) different numbers, all of the same colour, that have a product with exactly 1000 different natural divisors.

CC302. Nikolas used construction paper to make a regular tetrahedron (a pyramid consisting of equilateral triangles). Then he cut it in some ingenious way, unfolded it and this resulted in the following Christmas tree-like shape consisting of three halves of a regular hexagon:



How did Nikolas do this?

CC303. Consider two convex polygons M and N with the following properties: polygon M has twice as many acute angles as polygon N has obtuse angles; polygon N has twice as many acute angles as polygon M has obtuse angles; each polygon has at least one acute angle; at least one of the polygons has a right angle.

a) Give an example of such polygons.

b) How many right angles can each of these polygons have? Find the complete set of all the possibilities and prove that no others exist.

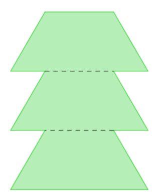
CC304. Consider a natural number n greater than 1 and not divisible by 10. Can the last digit of n and second last digit of n^4 (that is, the digit in the tens position) be of the same parity?

CC305. Can you arrange n identical cubes in such a way that each cube has exactly three neighbours (cubes are considered to be neighbours if they have a common face)? Solve the problem for $n = 2016, 2017$ and 2018.

. . . .

CC301. On a coloré tous les entiers strictement positifs en utilisant 100 couleurs distinctes. Démontrer qu'il est possible de trouver au moins deux nombres différents, tous de la même couleur, dont le produit admet exactement 1000 diviseurs différents.

CC302. Nicolas a utilisé du papier de bricolage pour construire un tétraèdre régulier (une pyramide dont les quatre faces sont des triangles équilatéraux). Il a ensuite découpé le tétraèdre de façon ingénieuse et l'a déplié de manière à obtenir la forme suivante composée de trois moitiés d'un hexagone régulier:



Comment Nicolas s'est-il pris?

CC303. On considère deux polygones convexes, M et N, qui satisfont aux conditions suivantes: le nombre d'angles aigus du polygone M est deux fois le nombre d'angles obtus du polygone N; le nombre d'angles aigus du polygone N est deux fois le nombre d'angles obtus du polygone M; chaque polygone a au moins un angle aigu; au moins un des polygones a un angle droit.

a) Donner un exemple de deux tels polygones.

b) Combien d'angles droits chacun de ces polygones peut-il avoir? Déterminer l'ensemble complet de toutes les possibilités et démontrer qu'il n'en existe aucune autre.

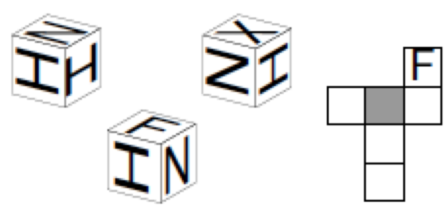
CC304. On considère un entier n supérieur à 1 qui n'est pas divisible par 10. Est-il possible que le chiffre des unités de n et le chiffre des dizaines de n^4 aient la même parité?

CC305. Est-il possible de placer n cubes identiques de manière que chaque cube ait exactement trois voisins (deux cubes sont voisins si un cube a une face qui touche au complet à une face de l'autre cube)? Résoudre le problème pour n égal à 2016, 2017 et 2018.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2017: 43(1), p. 5–7.

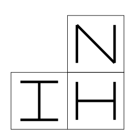
CC251. The six faces of a cube are labeled F, H, I, N, X and Z. Three views of the labelled cube are shown. Note that the H and the N on the die are indistinguishable from the rotated I and Z, respectively. The cube is then unfolded to form the lattice shown, with F shown upright. What letter should be drawn upright on the shaded square?



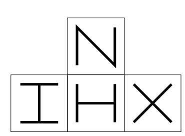
Originally Problem 10, Junior Round part A, of the 2015 BC Secondary Math Contest.

We received two complete solutions. We present a solution loosely based on that by the Missouri State University Problem Solving Group.

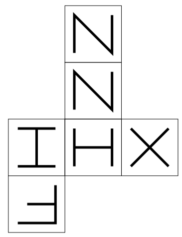
We start with the view of the cube containing both the H and the I. Partially unfolded, it looks as follows:



Now we try to align the second view of the cube, containing the X. The face labelled H or I in this view must be one of the two faces in the bottom row in the figure above. Of the four possible ways to overlay them, only one works with the remaining faces:



The third view, containing the F, has only one way to match with this, which is



By rotating the perspective by 180° we see that the answer to the question is H.

CC252. There are ten coins, each blank on one side and numbered on the other side with numbers 1 through 10. All ten coins are tossed and the sum of the numbers landing face up is calculated. What is the probability that this sum is at least 45?

Originally Problem 4, Senior Round part B, of the 2014 BC Secondary Math Contest.

We received 4 solutions; 1 was correct and complete. Two of the four were very close, but forgot to count one outcome in their arguments. We present the solution by Ivko Dimitrić.

Let us label each coin by the number that appears on its numbered side (the one that is not blank). We compute the required probability as the ratio of the number of favourable outcomes (when the sum of the numbers that turn up is at least 45) to the total number of all possible outcomes, each outcome being a string of ten symbols composed of numbers and blanks. The number of all outcomes is 2^{10}, since every one of the ten coins can turn blank side up or the numbered side up when the coins are flipped (two possibilities for each coin, independently of one another).

The sum of all ten numbers is $10 \cdot 11/2 = 55$ and a favourable outcome is any one for which the sum of numbers that turn up is at least 45, that is, when the sum of numbers that did not turn up is at most 10. Thus, no more than 4 blank sides could possibly turn up when the coins are flipped since, otherwise, the largest sum obtained would be less than $55-(1+2+3+4) = 45$. Therefore, we make our count of favourable outcomes for each case, depending on the number $k \in \{0, 1, 2, 3, 4\}$ of blank faces that turn up.

(0) If there is no blank face turned up and all the numbers appear, the sum is 55, so such (single) outcome is favourable,

(1) If there is exactly one blank that turned up, the outcome will be always favourable since the other side of that coin with the blank that turned up would have a number on it that is at most 10, so that the sum of all other numbers that turned up is at least $55 - 10 = 45$. The number of these outcomes is 10, since the blank side can turn up with any (but only one) of the ten coins.

(2) If there are exactly two blanks up, the outcome is favourable when the sum of two numbers on downsides of these coins is at most 10. That happens for the following 20 pairs in order:

$$(1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),\ (1,7),\ (1,8),\ (1,9),\ (2,3),\ (2,4),$$
$$(2,5),\ (2,6),\ (2,7),\ (2,8),\ (3,4),\ (3,5),\ (3,6),\ (3,7),\ (4,5),\ (4,6),$$

where each number also refers to the label of the coin in which the blank turned up with that number on the down side.Thus we have 20 favourable outcomes of this kind.

(3) If there are three blank sides up, for a favourable outcome to occur the sum of three numbers on downsides should be at most 10. There are 11 such triples:

$$(1,2,3),\ (1,2,4),\ (1,2,5),\ (1,2,6),\ (1,2,7),$$
$$(1,3,4),\ (1,3,5),\ (1,3,6),\ (1,4,5),\ \ (2,3,4),\ \ (2,3,5).$$

(4) Finally, if there are four blanks that turn up, there is only one favourable outcome, namely when the numbers on down sides are $(1,2,3,4)$ in the same numbered coins. Any other quadruple would have the sum of numbers that did not turn up greater than 10, causing the sum of numbers that did turn up less then $55-10=45$.

Hence, we have $1+10+20+11+1=43$ favourable outcomes out of total of $2^{10}=1024$ possible outcomes so that the required probability equals

$$\frac{43}{1024} \approx 0.042.$$

CC253. Let $A(n)$ represent the number of ways n pennies can be arranged in any number of rows, where each row starts at the same position as the row below it and has fewer pennies than the row below it. For example, $A(6)=4$, as shown below:

1. Show that $A(9)=8$.

2. Find the smallest number k which is not equal to $A(n)$ for any n.

Originally Problem 5, Junior Round part B, of the 2015 BC Secondary Math Contest.

We received 3 correct solutions. Solution by Dan Daniel.

Let $\Lambda(n)$ denote the set of allowable arrangements of n pennies and $|\Lambda(n)|$ its size. Then

1. We have $\Lambda(9) = \{(9),(8,1),(7,2),(6,3),(5,4),(6,2,1),(5,3,1),(4,3,2)\}$, so $A(9) = |\Lambda(9)| = 8$.

2. Easy calculation gives the following values for the first several values of $A(n)$.

n	1	2	3	4	5	6	7	8	9
$A(n)$	1	1	2	2	3	4	5	6	8

Hence $A(n) \neq 7$ for $n \leq 9$. If $(k_1, k_2, ...) \in \Lambda(n)$, then $(k_1+1, k_2, ...) \in \Lambda(n+1)$ so that $|\Lambda(n+1)| \geq |\Lambda(n)|$. Hence $A(n) \geq 8$ for $n \geq 10$, and $A(n) \neq 7$ for any n.

CC254. Hayden has a lock with a combination consisting of two 8s separated by eight digits, two 7s separated by seven digits, all the way down to two 1s separated by one digit. Unfortunately, Hayden spilled coffee on the paper that the combination was written on, and all that can be read of the combination is

$$* * *584 * * * * * * * * * *$$

Determine all the possible combinations of the lock.

Originally Problem 4, Junior Round part B, of the 2015 BC Secondary Math Contest.

We received three correct solutions and one incorrect solution. We present the solution of Ivko Dimitrić.

There are only two possible combinations:

$$4635843765121827 \quad \text{and} \quad 1615847365432872.$$

There are 16 places, numbered 1 through 16 from left to right, for 16 digits $d_1, d_2, \ldots, d_{16}$ of the combination lock of which three digits are known, $d_4 = 5, d_5 = 8, d_6 = 4$. For each digit that has been determined, the other one in the pair of the same digits may occur to the left or to the right of the known digit (provided there is room to place that digit) separated by the required number of places. Since $d_5 = 8$, the other digit 8 must be to the right of the given one (there is no room to the left) separated by eight places, so $d_{14} = 8$. Also, $d_4 = 5$ is given in the fourth place so the other digit five, separated by five places from the first one also has to be to the right, i.e. $d_{10} = 5$. One digit 4 is given in the sixth place, $d_6 = 4$, so the other digit 4 is either in the first position or in the 11th position, that is $d_1 = 4$ or $d_{11} = 4$.

The digit in the second position could be one of the remaining digits 1, 2, 3, 6, 7. However, all other digits, except 6, cannot occupy that position since there is no room to the right (or left) for the other digit of the pair to be placed. For example, if $d_2 = 7$ the other digit 7 would have to be to the right of the first one separated by 7 places i. e. in the position occupied by digit $d_{10} = 5$. Similarly, the impossibility for digits 1, 2, 3 to be placed in the second position is also obvious. Thus $d_2 = 6$ and consequently $d_9 = 6$.

Regarding the possibilities for the position of second digit 4, we distinguish between two possible arrangements:

1. If $d_1 = 4$, then digits 4, 5, 6, and 8 have been used to create the following portion of the combination:

$$46 * 584 * *65 * * * 8 * *.$$

The third digit could be only 3 or 7 on account of the available room to place the other digit of the pair. If $d_3 = 3$ then $d_7 = 3$ and $d_8 = d_{16} = 7$, producing $4635843765 * * * 8 * 7$. There remain now only two pairs of digits 1 and 2 to be placed. Digit d_{11} cannot be 2, otherwise the position for the second digit 2 would be blocked by 8 on the right and 7 on the left. Thus $d_{11} = d_{13} = 1$ and finally $d_{12} = d_{15} = 2$ to complete one of the possible combinations:

$$4635843765121827.$$

If $d_3 = 7$, then $d_{11} = 7$, which means $d_8 = d_{12} = 3$ or $d_{12} = d_{16} = 3$. Either way, we cannot complete the combination.

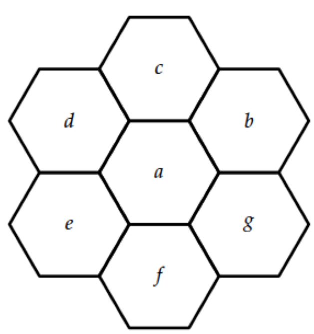
2. If $d_{11} = 4$ then the recovered portion of the combination reads as

$$*6 * 584 * *654 * * * * * .$$

In this arrangement, we must have $d_1 = 1$ since for the other possibilities 2, 3, 7 for the first position, the place for the second digit of the pair would have been already taken by other digits. Hence $d_3 = 1$. Digit d_7 cannot be 2 or 3, otherwise there would be no room to the right (or left) to place the second digit of the same pair. Thus $d_7 = 7$ and then $d_{15} = 7$. Then $d_8 = 3$ (that position cannot be taken by 2 since there would be no room for the other 2) and hence $d_{12} = 3$. The remaining two places are taken by digits 2, $d_{13} = d_{16} = 2$, for the other possible combination:

$$1615847365432872.$$

CC255. Antonino is instructed to colour the honeycomb pattern shown, which is made up of labelled hexagonal cells:



If two cells share a common side, they are to be coloured with different colours. Antonino has four colours available. Determine the number of ways he can colour the honeycomb, where two colourings are different if there is at least one cell that is a different colour in the two colourings.

Originally Problem 10, Senior Round part A, of the 2014 BC Secondary Math Contest.

We received 3 solutions, all correct and complete. We present 2 solutions.

Solution 1, by Ivko Dimitrić.

There are 264 different colourings.

Denote four colours by letters W, X, Y and Z. Once the colour for the central cell a is chosen, say W (and we write $a = W$) then the remaining 6 cells are to be coloured by three remaining colours (X, Y, Z) according to the rules. The pattern cannot be coloured just by two colours since for three neighbouring cells a, b, c one would need three different colours. Three colours would suffice where the cells b, d, f must be one colour (other than the colour of a) and cells c, e, g the third colour. Given the colour of a (4 choices for that) we can choose two additional colours for cells $b - g$ out of the remaining three in $\binom{3}{2} = 3$ ways so that each can be used for cells b, d, f (2 ways for that) and the remaining one for cells c, e, g. Therefore, we get $4 \cdot 3 \cdot 2 = 4! = 24$ ways to colour the cells if only three out of four colours are used.

Now consider the number of ways if all 4 colours are used. Once the choice for the colour of a has been made (4 choices), for example $a = W$, there are choices for diagonal cells b and e that involve same or different colours.

1. There are three ways to colour b and e by the same colour X, Y or Z. If that colour is X then according to the rule that no two neighbours are the same colour, there are two choices for colouring f and g and independently two choices for colouring d and e, namely $de, fg \in \{YZ, ZY\}$ which involves 4 combinations. That makes $4 \cdot 3 = 12$ possibilities for such an arrangement.

2. If different colours are used for b and e, then there are 6 possibilities

$$be = XY, YX, XZ, ZX, YZ, ZY$$

for the choice of these colours (as above we assume that the colour of a has been chosen to be W.) Let $b = X$, $e = Y$. Then the possible (mutually different) colours for f and d are X and Z, in any order, and colours for e and g are Y and Z. This makes 3 choices for the pair $fg \in \{XY, XZ, ZY\}$ and the same three choices independently for the pair de. That makes $3 \cdot 3 = 9$ possibilities for colouring d, e, f, g once the colours for b and e have been chosen in this arrangement. That amounts to $9 \cdot 6 = 54$ different colourings of cells $b - g$ in this arrangement once the colour for cell a has been selected. Add the previous count in 1. to the count in 2. to get $12 + 54 = 66$ ways to do the colouring once the colour of cell a has been fixed. If we now vary the colour of a (among 4 choices) that will produce $4 \cdot 66 = 264$ ways in total.

Solution 2, by Richard Hess.

The diagrams below demonstrate 11 basic ways to distribute the colours. Each of these admits to 24 cases given by permuting the four colours. Thus there are a total of 264 ways to colour the pattern.