

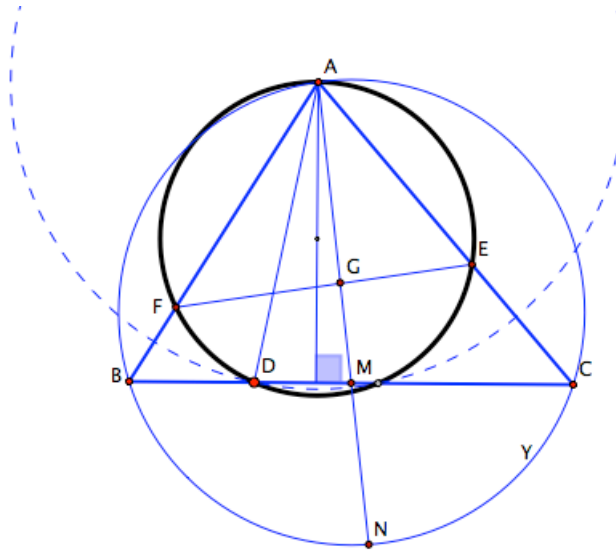
# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

*Statements of the problems in this section originally appear in 2016: 42(9), p. 400–404.*

**4181.** *Proposed by Marius Stănean.*

Let  $D \in BC$  be the foot of the  $A$ -symmedian of triangle  $ABC$  with centroid  $G$  (where the  $A$ -symmedian is the reflection of the median at  $A$  in the bisector of angle  $A$ ). The circle passing through  $A$ ,  $D$  and tangent to the line parallel to  $BC$  passing through  $A$  intersects sides  $AB$  and  $AC$  at  $F$  and  $E$ , respectively. If  $3AD^2 = AB^2 + AC^2$ , prove that  $G$  lies on  $EF$ .



*We received 6 solutions, all of which were correct; we feature the similar solutions by Peter Woo and the proposer, combined and expanded upon by the editor.*

It turns out that properties of the symmedian are irrelevant to the problem — we shall prove a stronger result, namely:

*Given a triangle  $ABC$  with centroid  $G$ , let  $D$  be a point of the line  $BC$ , and define  $E$  and  $F$  to be the points where the circle through  $A$  and  $D$  that is tangent to the line parallel to  $BC$  passing through  $A$  intersects sides  $AC$  and  $AB$ . Then  $G$  lies on  $EF$  if and only if the length of  $AD$  satisfies  $3AD^2 = AB^2 + AC^2$ .*

Denote by  $\mathcal{J}$  inversion in the circle with center  $A$  and radius  $AD$ . Because the given circle  $AEF$  is defined to be symmetric about the line through  $A$  that is

perpendicular to  $BC$ , while  $D \in BC$  is fixed by the inversion,  $\mathcal{J}$  must interchange the line  $BC$  with the circle  $AEF$ . Specifically,  $C$  and  $E$  are interchanged by the inversion as are  $B$  and  $F$ . Consequently,  $\mathcal{J}$  interchanges the line  $EF$  with the circumcircle of  $\triangle ABC$ , call it  $\gamma$ . Denoting by  $M$  the midpoint of  $BC$  and by  $N$  the point where the extension  $AM$  of the median intersects  $\gamma$ , we have, in particular, the image  $N'$  of  $N$  under  $\mathcal{J}$  always lies on  $EF$ . Consequently, we are required to prove that  $N'$  coincides with the centroid  $G$  if and only if  $3AD^2 = AB^2 + AC^2$ . To that end we introduce the standard notation  $a, b, c$  for the lengths of sides  $BC, AC, AB$ , and  $m = AM$  for the length of the median. As a consequence of Stewart's theorem we have

$$4m^2 = 2b^2 + 2c^2 - a^2.$$

Moreover, the chords of  $\gamma$  that intersect at  $M$  yield  $AM \cdot MN = BM \cdot MC = \frac{a^2}{4}$ , or

$$MN = \frac{a^2}{4m}.$$

Thus,

$$AN = m + MN = \frac{4m^2 + a^2}{4m} = \frac{b^2 + c^2}{2m}.$$

Because the circle of inversion has radius  $AD$ , the inverse  $N'$  of  $N$  satisfies  $AN' = \frac{AD^2}{AN} = \frac{2mAD^2}{b^2 + c^2}$ . But the centroid of the given triangle is the point between  $A$  and  $N$  that satisfies  $AG = \frac{2}{3}m$ . Thus  $N' = G$  if and only if  $\frac{2}{3}m = \frac{2mAD^2}{b^2 + c^2}$ , which immediately reduces to the required equation.

*Editor's comments.* The other four solvers began with coordinates for the foot of the symmedian, and thereby failed to discover the more general result. Their computations suggest that there *might* exist a one-parameter family of triangles  $ABC$  with a given side  $BC$  for which  $D$  is the foot of the symmedian and  $G \in EF$ ; this conjecture is supported (but, of course, not proved) by computer graphics. By analyzing isosceles triangles it is easily seen that there exist triangles for which  $G \in EF$  even though  $AD$  is not the symmedian (which for isosceles triangles coincides with the perpendicular bisector of  $BC$ ). Indeed, we have seen that so long as  $\sqrt{\frac{b^2 + c^2}{3}}$  exceeds the altitude, there will be two positions of  $D$  on  $BC$  for which  $G \in EF$ .

#### 4182. Proposed by Michel Bataille.

Let  $F_m$  be the  $m$ th Fibonacci number (defined by  $F_0 = 0, F_1 = 1, F_{m+2} = F_{m+1} + F_m$  for all integers  $m \geq 0$ ) and let  $n$  be a positive integer. For  $k = 1, 2, \dots, n$ , let

$$U_k = \frac{k}{F_{n+1-k}F_{n+3-k}} + (-1)^{k+1} \frac{2F_k}{F_{k+2}}.$$

Prove that  $|U_1 + U_2 + \dots + U_n - n|$  is the quotient of two Fibonacci numbers.

*We received 6 solutions, all of which were correct. We present the solution by the proposer.*

For positive integer  $n$ , let  $S_n = U_1 + U_2 + \cdots + U_n$ . Since we will use induction on  $n$  and  $U_k$  is dependent on  $n$  as well as  $k$ , we express  $S_n$  in a form that makes its dependence on  $n$  more amenable to managing the induction step. Note that

$$S_1 = 1/F_1F_3 + 2F_1/F_3 = 3/2 = 1 + F_1/F_3,$$

and, for  $n \geq 2$ ,

$$\begin{aligned} S_n &= \sum_{k=1}^n \frac{k}{F_{n+1-k}F_{n+3-k}} + \sum_{k=1}^n (-1)^{k+1} \frac{2F_k}{F_{k+2}} \\ &= \sum_{k=1}^n \frac{n+1-k}{F_kF_{k+2}} + \sum_{k=1}^n (-1)^{k+1} \frac{2F_k}{F_{k+2}} \\ &= \sum_{k=1}^n \frac{1}{F_kF_{k+2}} + \sum_{k=1}^{n-1} \frac{n-k}{F_kF_{k+2}} + \sum_{k=1}^n (-1)^{k+1} \frac{2F_k}{F_{k+2}} \\ &= S_{n-1} + \sum_{k=1}^n \frac{1}{F_kF_{k+2}} + (-1)^{n+1} \frac{2F_n}{F_{n+2}}. \end{aligned}$$

Since

$$\begin{aligned} \sum_{k=1}^n \frac{1}{F_kF_{k+2}} &= \sum_{k=1}^n \frac{F_{k+2} - F_k}{F_kF_{k+1}F_{k+2}} = \sum_{k=1}^n \left( \frac{1}{F_kF_{k+1}} - \frac{1}{F_{k+1}F_{k+2}} \right) \\ &= 1 - \frac{1}{F_{n+1}F_{n+2}}, \end{aligned}$$

$$S_n = S_{n-1} + 1 - \frac{1}{F_{n+1}F_{n+2}} + (-1)^{n+1} \frac{2F_n}{F_{n+2}}.$$

We prove that

$$S_n - n = (-1)^{n+1} \frac{F_n}{F_{n+2}}$$

for  $n \geq 1$ . This holds for  $n = 1$ . Suppose it holds for  $n = m - 1$ . Then

$$\begin{aligned} S_m - m &= S_{m-1} - (m-1) - \frac{1}{F_{m+1}F_{m+2}} + (-1)^{m+1} \frac{2F_m}{F_{m+2}} \\ &= (-1)^m \frac{F_{m-1}}{F_{m+1}} - \frac{1}{F_{m+1}F_{m+2}} + (-1)^{m+1} \frac{2F_m}{F_{m+2}} \\ &= \frac{(-1)^{m+1} [F_m F_{m+1} + (F_m F_{m+1} - F_{m-1} F_{m+2})] - 1}{F_{m+1} F_{m+2}} \\ &= \frac{(-1)^{m+1} [F_m F_{m+1} + (-1)^{m+1}] - 1}{F_{m+1} F_{m+2}} = (-1)^{m+1} \frac{F_m}{F_{m+2}}, \end{aligned}$$

as desired.

Therefore,  $|S_n - n|$  is equal to  $F_n/F_{n+2}$ , the quotient of two Fibonacci numbers.

*Editor's comments.* Joel Schlosberg gave a direct argument, by obtaining a double sum:

$$\begin{aligned} \sum_{j=1}^n \frac{n+1-j}{F_j F_{j+2}} &= \sum_{k=1}^n \sum_{j=1}^k \frac{1}{F_j F_{j+2}} = \sum_{k=1}^n \left( \frac{1}{F_1 F_2} - \frac{1}{F_{k+1} F_{k+2}} \right) \\ &= n - \sum_{k=1}^n \frac{1}{F_{k+1} F_{k+2}}. \end{aligned}$$

Then  $U_1 + \cdots + U_n - n$  can be evaluated similarly to the foregoing solution.

**4183.** *Proposed by Lorian Saceanu and Leonard Giugiuc.*

Let  $ABC$  be a non obtuse triangle with orthocenter  $H$  and circumradius  $R$ . Prove that

$$(3\sqrt{3} - 4) \cdot AH \cdot BH \cdot CH \geq abc - 4R^3$$

and determine when the equality holds.

*We received 3 correct solutions and one incorrect solution. We present the solution by Leonard Giugiuc, submitted independently.*

This proof is based on Blundon's theorem, which states that in any triangle,

$$s \leq 2R + (3\sqrt{3} - 4)r,$$

where the notations are as customary. Equality holds if and only if the triangle is equilateral. For a reference, see

[https://www.emis.de/journals/JIPAM/images/220\\_08\\_JIPAM/220\\_08.pdf](https://www.emis.de/journals/JIPAM/images/220_08_JIPAM/220_08.pdf).

Since triangle  $ABC$  is non obtuse, we have

$$AH = 2R \cos A, \quad BH = 2R \cos B, \quad CH = 2R \cos C.$$

Also, by the law of sines,

$$a = 2R \sin A, \quad b = 2R \sin B, \quad c = 2R \sin C.$$

The required inequality is thus equivalent to

$$2(\sqrt{3} - 4 \cos A \cos B \cos C) \geq 2 \sin A \sin B \sin C - 1.$$

*Case 1.* Triangle  $ABC$  is right-angled. Without loss of generality,  $A = \frac{\pi}{2}$ . The required inequality is equivalent to

$$1 \geq 2 \sin A \sin B \sin C,$$

implying that  $1 \geq \sin 2B$ , which is true.

*Case 2.* Triangle  $ABC$  is acute-angled. Let  $DEF$  be the orthic triangle of  $ABC$ ; that is,  $D$ ,  $E$ , and  $F$  are the endpoints of the altitudes of  $ABC$ . Standard facts for orthic triangles give

$$R_{DEF} = \frac{R}{2}, \quad s_{DEF} = 2R \sin A \sin B \sin C, \quad \text{and} \quad r_{DEF} = 2R \cos A \cos B \cos C.$$

By Blundon's Theorem,

$$s_{DEF} \leq 2R_{DEF} + (3\sqrt{3} - 4)r_{DEF},$$

or equivalently

$$2(3\sqrt{3} - 4) \cos A \cos B \cos C \geq 2 \sin A \sin B \sin C - 1.$$

Equality holds if and only if  $DEF$  is equilateral, i.e., if and only if  $ABC$  is equilateral.

**4184.** *Proposed by Mihaela Berindeanu.*

Evaluate the following integral

$$\int_{32}^{63} \frac{\ln 2016x}{x^2 + 2016} dx.$$

*We received 12 submissions of which 8 were correct and complete. We present 2 solutions.*

*Solution 1, by Michel Bataille.*

Let  $I$  be the given integral. We show that

$$I = \frac{3 \ln(2016)}{2\sqrt{2016}} \left( 2 \arctan \left( \sqrt{\frac{63}{32}} \right) - \frac{\pi}{2} \right).$$

We shall use the following lemma: if  $\alpha > 0$ , then

$$\int_{\alpha}^{1/\alpha} \frac{\ln x}{x^2 + 1} dx = 0.$$

*Proof.* The change of variables  $x = \frac{1}{u}$ ,  $dx = \frac{-du}{u^2}$  gives

$$\int_{\alpha}^{1/\alpha} \frac{\ln x}{x^2 + 1} dx = \int_{1/\alpha}^{\alpha} \frac{-\ln u}{1 + \frac{1}{u^2}} \cdot \frac{-du}{u^2} = - \int_{\alpha}^{1/\alpha} \frac{\ln u}{u^2 + 1} du$$

and the result follows. □

Now,  $I = I_1 + I_2$  with

$$I_1 = \ln(2016) \int_{32}^{63} \frac{dx}{x^2 + 2016} = \frac{\ln(2016)}{\sqrt{2016}} \left( \arctan \left( \sqrt{\frac{63}{32}} \right) - \arctan \left( \sqrt{\frac{32}{63}} \right) \right)$$

and

$$I_2 = \int_{32}^{63} \frac{\ln x}{x^2 + 2016} dx = \frac{\ln(2016)}{2\sqrt{2016}} \int_{\sqrt{\frac{32}{63}}}^{\sqrt{\frac{63}{32}}} \frac{dy}{1+y^2} + \frac{1}{\sqrt{2016}} \int_{\sqrt{\frac{32}{63}}}^{\sqrt{\frac{63}{32}}} \frac{\ln y}{1+y^2} dy$$

with the change of variables  $x = y\sqrt{2016}$ .

Using the lemma,

$$\int_{\sqrt{\frac{32}{63}}}^{\sqrt{\frac{63}{32}}} \frac{\ln y}{1+y^2} dy = 0,$$

hence we have

$$I_2 = \frac{\ln(2016)}{2\sqrt{2016}} \left( \arctan \left( \sqrt{\frac{63}{32}} \right) - \arctan \left( \sqrt{\frac{32}{63}} \right) \right).$$

In conclusion,

$$\begin{aligned} I &= \frac{3\ln(2016)}{2\sqrt{2016}} \left( \arctan \left( \sqrt{\frac{63}{32}} \right) - \arctan \left( \sqrt{\frac{32}{63}} \right) \right) \\ &= \frac{3\ln(2016)}{2\sqrt{2016}} \left( 2\arctan \left( \sqrt{\frac{63}{32}} \right) - \frac{\pi}{2} \right), \end{aligned}$$

where the latter is true because  $\arctan a + \arctan(1/a) = \frac{\pi}{2}$  when  $a > 0$ .

*Solution 2, by Leonard Giugiuc.*

Let  $a$  and  $b$  be real numbers with  $b > a > 0$ . We will evaluate

$$I = \int_a^b \frac{\ln abx}{x^2 + ab} dx.$$

Making the substitution  $\frac{ab}{x} \rightarrow x$ , we get

$$I = \int_a^b \frac{\frac{ab}{x^2} \ln \frac{(ab)^2}{x}}{\frac{ab}{x^2}(x^2 + ab)} dx = \int_a^b \frac{\ln \frac{(ab)^2}{x}}{x^2 + ab} dx.$$

So,

$$\begin{aligned} 2I &= \int_a^b \frac{\ln abx}{x^2 + ab} dx + \int_a^b \frac{\ln \frac{(ab)^2}{x}}{x^2 + ab} dx \\ &= 3\ln ab \int_a^b \frac{1}{x^2 + ab} dx \\ &= \frac{3\ln ab}{\sqrt{ab}} \left( \arctan \sqrt{\frac{b}{a}} - \arctan \sqrt{\frac{a}{b}} \right) \\ &= \frac{3\ln ab}{\sqrt{ab}} \arctan \frac{b-a}{2\sqrt{ab}}. \end{aligned}$$

Setting  $a = 32$  and  $b = 63$ , we obtain

$$\int_{32}^{63} \frac{\ln 2016x}{x^2 + 2016} dx = \frac{3 \ln 2016}{2\sqrt{2016}} \arctan \frac{31}{2\sqrt{2016}}.$$

**4185.** *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Prove that for any positive real numbers  $a, b, c$  and  $k$ , we have

$$\left[ \frac{a^{k-1}(a^2 + bc)}{(b+c)^{k+1}} \right]^{\frac{1}{k}} + \left[ \frac{b^{k-1}(b^2 + ca)}{(c+a)^{k+1}} \right]^{\frac{1}{k}} + \left[ \frac{c^{k-1}(c^2 + ab)}{(a+b)^{k+1}} \right]^{\frac{1}{k}} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

We received 3 correct solutions. We present the solution by S efket Arslanagi c.

Without loss of generality we will take  $a \geq b \geq c$ ; it follows then that

$$\frac{a}{b+c} \geq \frac{b}{c+a} \geq \frac{c}{a+b}$$

and

$$\frac{a^2 + bc}{a(b+c)} \geq \frac{b^2 + ca}{b(c+a)} \geq \frac{c^2 + ab}{c(a+b)},$$

and we have by Chebyshev's inequality,

$$\sum_{\text{cyc}} \frac{a}{b+c} \left( \frac{a^2 + bc}{a(b+c)} \right)^{1/k} \geq \frac{1}{3} \left( \sum_{\text{cyc}} \frac{a}{b+c} \right) \left( \sum_{\text{cyc}} \left( \frac{a^2 + bc}{a(b+c)} \right)^{1/k} \right). \quad (1)$$

We now make use of the inequality

$$(a^2 + bc)(b^2 + ca)(c^2 + ab) \geq abc(a+b)(b+c)(c+a),$$

which holds since, upon expansion, it becomes the sum of the Muirhead inequalities  $(3, 3, 0) \succcurlyeq (3, 2, 1)$  and  $(4, 1, 1) \succcurlyeq (3, 2, 1)$ . Thus, by the AM-GM inequality,

$$\sum_{\text{cyc}} \left( \frac{a^2 + bc}{a(b+c)} \right)^{1/k} \geq 3 \cdot \sqrt[3]{\left[ \frac{(a^2 + bc)(b^2 + ca)(c^2 + ab)}{a(b+c)(b+c)(c+a)} \right]^{1/k}} \geq 3. \quad (2)$$

The claimed inequality now follows from (1) and (2). Equality holds if and only if  $a = b = c$ .

**4186.** *Proposed by Florin Stanescu.*

Let  $f, g : [0, 1] \rightarrow [0, \infty)$ ,  $f(0) = g(0) = 0$  be two continuous functions such that  $f$  is convex and  $g$  is concave. If  $h : [0, 1] \rightarrow \mathbb{R}$  is an increasing function, show that

$$\int_0^1 g(x)h(x)dx \cdot \int_0^1 f(x)dx \leq \int_0^1 g(x)dx \cdot \int_0^1 h(x)f(x)dx.$$

We received 2 correct solutions and feature the one by Leonard Giugiuc.

If  $\int_0^1 f(x)dx = 0$ , then, since  $f$  is continuous and  $f(x) \geq 0, \forall x \in [0, 1]$ ,  $f \equiv 0$ , and hence the inequality is proved. The same conclusion holds if  $\int_0^1 g(x)dx = 0$ . So we assume that

$$I = \int_0^1 f(x)dx > 0 \text{ and } J = \int_0^1 g(x)dx > 0.$$

The required inequality is equivalent to

$$\int_0^1 h(x) \left( \frac{f(x)}{I} - \frac{g(x)}{J} \right) dx \geq 0.$$

Define the function  $\phi : [0, 1] \rightarrow \mathbb{R}$  as  $\phi(x) = \frac{f(x)}{I} - \frac{g(x)}{J}, \forall x \in [0, 1]$ . Clearly,  $\phi$  is continuous and convex. Moreover,

$$\int_0^1 \phi(x) = \frac{1}{I} \int_0^1 f(x) - \frac{1}{J} \int_0^1 g(x) = 1 - 1 = 0,$$

and  $\phi(0) = 0$ . Now, the required inequality is equivalent to  $\int_0^1 h(x)\phi(x)dx \geq 0$ .

If  $\phi$  is increasing on  $[0, 1]$ , then since  $\phi(0) = 0$  and  $\int_0^1 \phi(x) = 0$ , we deduce  $\phi \equiv 0$  and we are done. A similar conclusion holds if  $\phi$  is decreasing. Otherwise, since  $\phi$  is convex, we know  $\exists t \in (0, 1)$  such that  $\phi$  is decreasing on  $[0, t]$  and  $\phi$  is increasing on  $[t, 1]$ . Thus it follows that  $\phi(1) \geq 0$  and that  $\exists c \in (0, 1)$  such that  $\phi(x) \leq 0 \forall x \in [0, c]$  and  $\phi(x) \geq 0 \forall x \in [c, 1]$ . Then  $\forall x \in [0, c]$ , we have  $h(x) - h(c) \leq 0$  and  $\phi(x) \leq 0$ , so that  $h(x)\phi(x) \geq h(c)\phi(x)$ . Similarly,  $\forall x \in [c, 1]$ , we have  $h(x) - h(c) \geq 0$  and  $\phi(x) \geq 0$ , so that  $h(x)\phi(x) \geq h(c)\phi(x)$ .

In conclusion,  $h(x)\phi(x) \geq h(c)\phi(x), \forall x \in [0, 1]$ , and hence

$$\int_0^1 h(x)\phi(x) \geq \int_0^1 h(c)\phi(x) = 0.$$

The proof is complete.

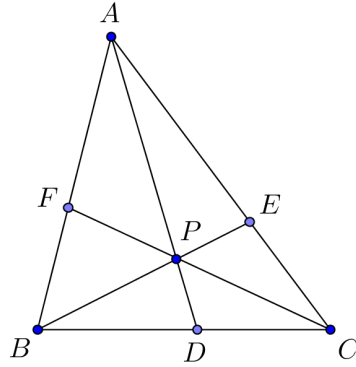
**4187.** Proposed by Avi Sigler and Moshe Stupel.

A point  $P$  inside triangle  $ABC$  divides the three cevians  $AD, BE, CF$  through  $P$  into segments whose harmonic means are

$$K_A = \frac{2AP \cdot PD}{AP + PD}, \quad K_B = \frac{2BP \cdot PE}{BP + PE}, \quad K_C = \frac{2CP \cdot PF}{CP + PF}.$$

Prove that these three harmonic means, each associated with a cevian, are proportional to the sines of the angles  $\angle CPE, \angle APF, \angle EPA$  formed between the other two cevians.





We received nine solutions, all of which were correct, and feature the solution whose similar versions came independently from Steven Chow and Oliver Geupel.

Let square brackets represent areas. From the relation

$$\frac{PD}{AD} = \frac{[PBC]}{[ABC]} = \frac{BP \cdot CP \sin \angle CPE}{2[ABC]},$$

we obtain

$$\sin \angle CPE = \frac{2[ABC] \cdot PD}{AD \cdot BP \cdot CP}.$$

Hence,

$$\frac{K_A}{\sin \angle CPE} = \frac{AP \cdot BP \cdot CP}{[ABC]}.$$

Similarly,

$$\frac{K_B}{\sin \angle APF} = \frac{K_C}{\sin \angle EPA} = \frac{AP \cdot BP \cdot CP}{[ABC]}.$$

As a consequence, the three harmonic means of the segments on one cevian are proportional to the sines of the angles formed between the other two cevians.

**4188.** *Proposed by Daniel Sitaru.*

Let  $0 < x < y < z < \frac{\pi}{2}$ . Prove that

$$(x + y) \sin z + (x - z) \sin y < (y + z) \sin x.$$

*There were eight correct solutions. Four of them gave essentially the solution below.*

Because  $\frac{\sin t}{t}$  is a decreasing function on  $(0, \pi/2)$ , we find that

$$x \sin y < y \sin x, \quad y \sin z < z \sin y, \quad \text{and} \quad x \sin z < z \sin x.$$

Therefore

$$\begin{aligned} & (y + z) \sin x - (x - z) \sin y - (x + y) \sin z \\ &= (y \sin x - x \sin y) + (z \sin y - y \sin z) + (z \sin x - x \sin z) > 0. \end{aligned}$$

**4189.** *Proposed by Mihaela Berindeanu.*

Prove that the equation

$$3y^2 = -2x^2 - 2z^2 + 5xy + 5yz - 4xz + 1$$

has infinitely many solutions in integers.

*We received 15 solutions, most of which solved the given equation. We present the solution by Arkady Alt.*

Rearrange and factor the given equation:

$$\begin{aligned} 2x^2 + 4xz + 2z^2 + 3y^2 - 5xy - 5yz &= 1 \Leftrightarrow \\ 2(x+z)^2 - 5y(x+z) + 3y^2 &= 1 \Leftrightarrow \\ (2(x+z) - 3y)(x+z-y) &= 1. \end{aligned}$$

Let  $w = x + z$ ; if  $x, z$  are integers then so is  $w$ . To find the integer solutions to the equation  $(2w - 3y)(w - y) = 1$  we consider two cases:

$$\begin{aligned} 2w - 3y = w - y = 1 \text{ OR} \\ 2w - 3y = w - y = -1. \end{aligned}$$

The first system of equations yields  $w = 2$  and  $y = 1$ , and the second  $w = -2$  and  $y = -1$ . From  $x + y = w$ , we get that the solutions to the original equation are  $\{(t, 1, 2 - t) : t \in \mathbb{Z}\} \cup \{(t, -1, -2 - t) : t \in \mathbb{Z}\}$ , showing that there are infinitely many solutions.

**4190.** *Proposed by Leonard Giugiuc.*

Let  $a, b, c, d$  and  $e$  be real numbers such that  $a + b + c + d + e = 20$  and  $a^2 + b^2 + c^2 + d^2 + e^2 = 100$ . Prove that

$$625 \leq abcd + abce + abde + acde + bcde \leq 945.$$

*Only the proposer supplied a solution.*

Suppose, wolog, that  $a \geq b \geq c \geq d \geq e$ . We first verify that  $a, b, c, d, e$  must be all nonnegative. Since

$$\begin{aligned} 4(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 & \\ = 3(a^2 + b^2 + c^2 + d^2) - 2(ab + ac + ad + bc + bd + be) & \\ = (a - b)^2 + (a - c)^2 + (a - d)^2 & \\ + (b - c)^2 + (b - d)^2 + (c - d)^2 \geq 0, & \end{aligned}$$

then

$$(a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2) \leq 400.$$

Thus  $a + b + c + d \leq 20$  and  $e \geq 0$ . Equality occurs if and only if  $(a, b, c, d, e) = (5, 5, 5, 5, 0)$ .

The foregoing inequality can be rewritten  $(20-e)^2 \leq 4(100-e^2)$ , whence  $0 \leq e \leq 8$ . An analogous argument certifies that  $a, b, c, d$  also lie in the closed interval  $[0, 8]$ .

Consider the quintic polynomial

$$p(x) = x^5 - 20x^4 + 150x^3 - 5qx^2 + 5rx - s,$$

whose roots are  $a, b, c, d, e$ , so that  $5q = \sum abc$ ,  $5r = \sum abcd$  and  $s = abcde$ . (We note that  $r \neq 0$ , for otherwise  $d = e = 0$  and  $300 = 3(a^2 + b^2 + c^2) \geq (a + b + c)^2 = 400$ , a contradiction.)

By Rolle's theorem,

$$p'(x) = 5(x^4 - 16x^3 + 90x^2 - 2qx + r)$$

has four real roots in the open interval  $(0, 8)$ , as does the function

$$f(x) = \frac{p'(x)}{5x} = x^3 - 16x^2 + 90x - 2q + \frac{r}{x}.$$

We have that  $f'(x) = x^{-2}g(x)$ , where  $g(x) = 3x^4 - 32x^3 + 90x^2 - r$ . By Rolle's theorem,  $f'(x)$  and so  $g(x)$  have three positive roots in  $(0, 8)$ . Since

$$g'(x) = 12x(x-3)(x-5),$$

the function  $g(x)$  is monotone on and has a root in each of the intervals  $(0, 3)$ ,  $(3, 5)$  and  $(5, 8)$ . We have  $0 > g(0) = -r$ ,  $0 \leq g(3) = 189 - r$  and  $0 \geq 125 - r$ . Therefore  $125 \leq r \leq 189$  and

$$625 \leq 5r = abcd + abce + abde + acde + bcde \leq 945.$$

*Editor's comment.* For each quintuple  $(a, b, c, d, e)$  satisfying the two conditions, there is an associate  $(8 - e, 8 - d, 8 - c, 8 - b, 8 - a)$  that also satisfies them. Two associate quintuples are  $(5, 5, 5, 5, 0)$  and  $(8, 3, 3, 3, 3)$ . In the former case,  $p(x) = x(x-5)^4$ ,  $p'(x) = 5(x-1)(x-5)^3$ ,  $g(x) = 3x^4 - 32x^3 + 90x^2 - 125 = (x-5)^2(x+1)(3x-5)$  and  $\sum abcd = 625$ . In the latter case,  $p(x) = (x-8)(x-3)^4$ ,  $p'(x) = 5(x-7)(x-3)^3$ ,  $f(x) = x^3 - 16x^2 + 90x - 216 + (189/x)$ ,  $g(x) = 3x^4 - 32x^3 + 90x^2 - 189 = (x-3)^2(3x^2 - 14x - 21)$ , and  $\sum abcd = 945$ .

There is one other quintuple of integers  $(7, 5, 4, 3, 1)$  satisfying the conditions which is its own associate. In this case  $\sum abcd = 809$ .

