OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2016 : 42(7), p. 297–298.

OC291. Let the integer $n \geq 2$, and $x_1, x_2, \cdots, x_n$ be positive real numbers such that $\sum_{i=1}^{n} x_i = 1$. Prove that

$$\left(\sum_{i=1}^{n} \frac{1}{1-x_i}\right)\left(\sum_{1\leq i<j\leq n} x_i x_j\right) \leq \frac{n}{2}.$$

Originally problem 3 from day 1 of the 2015 China Western National Olympiad.

We received 9 correct submissions. We present the solution by Michel Bataille.

Let

$$X = \sum_{1\leq i<j\leq n} x_i x_j.$$

From

$$\begin{aligned} X &= x_1(x_2 + \cdots + x_n) + \sum_{2\leq i<j\leq n} x_i x_j \\ &= x_1(1-x_1) + \sum_{2\leq i<j\leq n} x_i x_j \end{aligned}$$

we deduce

$$\frac{X}{1-x_1} = x_1 + \frac{\sum_{2\leq i<j\leq n} x_i x_j}{x_2 + \cdots + x_n} = x_1 + \frac{\sum_{2\leq i<j\leq n} x_i x_j}{1-x_1}. \tag{1}$$

But we have

$$\begin{aligned} \sum_{2\leq i<j\leq n} x_i x_j &= \frac{1}{2}\left((x_2+\cdots+x_n)^2 - (x_2^2+\cdots+x_n^2)\right) \\ &\leq \frac{1}{2}\left((x_2+\cdots+x_n)^2 - \frac{1}{n-1}(x_2+\cdots+x_n)^2\right), \end{aligned}$$

where the inequality follows from

$$(n-1)(x_2^2+\cdots+x_n^2) \geq (x_2+\cdots+x_n)^2$$

(by the Cauchy-Schwarz inequality).

Thus we obtain

$$\sum_{2\leq i<j\leq n} x_i x_j \leq \frac{n-2}{2(n-1)} \cdot (x_2+\cdots+x_n)^2 = \frac{(n-2)(1-x_1)^2}{2(n-1)}$$

and from (1)

$$\frac{X}{1-x_1} \le x_1 + \frac{(n-2)(1-x_1)}{2(n-1)}.$$

In the same way, with obvious changes, we get

$$\begin{aligned}\frac{X}{1-x_2} &\le x_2 + \frac{(n-2)(1-x_2)}{2(n-1)},\\ \frac{X}{1-x_3} &\le x_3 + \frac{(n-2)(1-x_3)}{2(n-1)}, \quad \ldots\\ \frac{X}{1-x_n} &\le x_n + \frac{(n-2)(1-x_n)}{2(n-1)}\end{aligned}$$

and so

$$\begin{aligned}\left(\sum_{i=1}^{n}\frac{1}{1-x_i}\right)\left(\sum_{1\le i<j\le n} x_i x_j\right) &= \sum_{i=1}^{n}\frac{X}{1-x_i}\\ &\le (x_1+x_2+\cdots+x_n) + \frac{n-2}{2(n-1)}\cdot\sum_{i=1}^{n}(1-x_i)\\ &= 1+\frac{n-2}{2(n-1)}(n-1) = \frac{n}{2}\end{aligned}$$

as required.

OC292. On the graph of a polynomial with integer coefficients, two points are chosen with integer coordinates. Prove that if the distance between them is an integer, then the segment that connects them is parallel to the horizontal axis.

Originally problem 1 from day 1 of the 2015 Spain Mathematical Olympiad.

We received 5 correct submissions. We present the solution by Steven Chow.

Let the x-axis be the horizontal axis. Let $f(x)$ be the polynomial function. Let a and b be the x-coordinates of the 2 points.

Since a, b, and the coefficients of $f(x)$ are integers, $a-b \mid f(a)-f(b)$.

Since the distance between the 2 points is an integer, from the Pythagorean Theorem, the following is a square number :

$$(a-b)^2 + (f(a)-f(b))^2 = (a-b)^2\left(1+\left(\frac{f(a)-f(b)}{a-b}\right)^2\right).$$

Therefore $1+\left(\frac{f(a)-f(b)}{a-b}\right)^2$ is a square number and $\frac{f(a)-f(b)}{a-b} = 0$ implies that $f(a)=f(b)$.

Thus, the segment that connects the 2 points is parallel to the horizontal axis.

OC293. You are given N such that $N \geq 3$. We call a set of N points on a plane acceptable if their abscissae are unique, and each of the points is coloured either red or blue. Let's say that a polynomial $P(x)$ divides a set of acceptable points either if there are no red dots above the graph of $P(x)$, and below, there are no blue dots, or if there are no blue dots above the graph of $P(x)$ and there are no red dots below. Keep in mind, dots of both colors can be present on the graph of $P(x)$ itself. For what least value of k is an arbitrary acceptable set of N points divisible by a polynomial of degree k?

Originally problem 4 of the 2015 All Russian Olympiad Grade 11.

We present the solution by Oliver Geupel. There were no other submissions.

The answer is $k = N - 2$.

Given any acceptable N-set, the interpolation polynomial of $N-1$ out of the N points is a polynomial of degree not greater than $N-2$ which divides that set. Hence $k \leq N-2$.

It is now sufficient to specify an acceptable N-set $\mathcal{S}_N$ such that every polynomial that divides $\mathcal{S}_N$ is of degree at least $N-2$. The blue points in our set $\mathcal{S}_N$ are $A_{-1}(-1,1)$, $A_0(0,-1)$, and $A_{2k}(2k,0)$ where $k = 1, 2, \ldots, \lfloor N/2 \rfloor - 1$. The red points in $\mathcal{S}_N$ are $A_{2k-1}(2k-1, 0)$ where $k = 1, 2, \ldots, \lceil N/2 \rceil - 1$. Those are in total $\lfloor N/2 \rfloor + \lceil N/2 \rceil = N$ points as required. Let $P(x)$ be a polynomial that divides $\mathcal{S}_N$. If $N = 3$, then $P(x)$ cannot be a constant, which shows that $k \geq 1 = N - 2$. It remains to consider $N \geq 4$.

We show that $\deg P(x) > 1$. Assume $\deg P(x) \leq 1$. Since the vertices of triangle $A_{-1}A_0A_2$ are of the same colour (blue), the graph of $P(x)$ cannot contain any interior point of the triangle. Thus, $P(x)$ cannot separate the interior red point A_1 from the blue vertices, which is a contradiction. Consequently, $\deg P(x) \geq 2$.

We have two cases : Either there are no red dots above the graph of $P(x)$ and no blue dots below, or there are no blue dots above the graph of $P(x)$ and no red dots below.

Let us first suppose that there are no red dots above the graph of $P(x)$ and no blue dots below. We then have $P(0) \leq P(1)$, $P(1) \geq P(2)$, $P(2) \leq P(3)$, continued with alternating order relations until either $P(N-3) \leq P(N-2)$ or $P(N-3) \geq P(N-2)$. By the Mean Value Theorem, there are real numbers $x_1 \in (0,1)$, $x_2 \in (1,2)$, $\ldots$, $x_{N-2} \in (N-3, N-2)$ such that $P'(x_1)$, $P'(x_2)$, $\ldots$, $P'(x_{N-2})$ have alternating signs. Hence, $\deg P(x) \geq N-2$.

It remains to consider the case where there are no blue dots above the graph of $P(x)$ and no red dots below that graph. We then readily see that $P(-1) \geq P(1)$, $P(1) \leq P(2)$, $P(2) \geq P(3)$, continued with alternating order relations until either $P(N-3) \leq P(N-2)$ or $P(N-3) \geq P(N-2)$. By the Mean Value Theorem, there are real numbers $x_1 \in (-1,1)$, $x_2 \in (1,2)$, $\ldots$, $x_{N-2} \in (N-3, N-2)$ such that $P'(x_1), P(x_2), \ldots, P'(x_{N-2})$ have alternating signs. Hence, $\deg P(x) \geq N-2$.

OC294. In given triangle $\triangle ABC$, difference between sizes of each pair of sides is at least $d > 0$. Let G and I be the centroid and incenter of $\triangle ABC$ and r be its inradius. Show that

$$|AIG| + |BIG| + |CIG| \geq \frac{2}{3}dr,$$

where $|XYZ|$ is the area of triangle $\triangle XYZ$.

Originally problem 5 from Round 3 Category A of the 2015 Czech and Slovak National Olympiad.

We received 2 correct submissions. We present the solution by Mohammed Aassila.

We use barycentric coordinates in triangle ABC. We know that $G = (1 : 1 : 1)$ and $I = (a : b : c)$ where $a = BC, b = CA$ and $c = AB$. We have :

$$[AIG] = \frac{[ABC]}{3(a+b+c)} \cdot \begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \frac{[ABC] \cdot |b-c|}{3(a+b+c)}.$$

Similar equations for $[BIG]$ and $[CIG]$ can be obtained in the same manner. Hence :

$$\begin{aligned} [AIG] + [BIG] + [CIG] &= \frac{[ABC] \cdot (|b-c| + |c-a| + |a-b|)}{3(a+b+c)} \\ &= \frac{r \cdot (|b-c| + |c-a| + |a-b|)}{6}. \end{aligned}$$

Assume without loss of generality that $a \leq b \leq c$. Since $b - a \geq d$ and $c - b \geq d$ we have that $c - a \geq 2d$. Thus

$$\frac{r \cdot (|b-c| + |c-a| + |a-b|)}{6} \geq \frac{2}{3} \cdot dr.$$

OC295. Let $\mathbb{N} = \{1, 2, 3, ...\}$ be the set of positive integers. Let $f : \mathbb{N} \to \mathbb{N}$ be a function that gives a positive integer value, to every positive integer. Suppose that f satisfies the following conditions :

$$f(1) = 1, \quad f(a + b + ab) = a + b + f(ab).$$

Find the value of $f(2015)$.

Originally problem 3 from day 1 of the 2015 Mexico National Olympiad.

We received 9 correct submissions. We present the solution by Missouri State University Problem Solving Group.

We will show that $f(n) = n$ for every $n \in \mathbb{N}$. Let $S = \{n \in \mathbb{N} : f(n) = n\}$. First we note that for any $a, b \in \mathbb{N}$, we have that

$$ab \in S \iff a + b + ab \in S$$

Since $2n+1 = 1+n+1\cdot n$, then for any $n \in \mathbb{N}$, we have

$$n \in S \iff 2n+1 \in S$$

By repeating this argument, we can then extend this to

$$n \in S \iff 2n+1 \in S \iff 4n+3 \in S \iff 8n+7 \in S$$

and so on. In particular, $1 \in S$ if and only if $8(1)+7 = 15 \in S$ and $2 \in S$ if and only if $8(2)+7 = 23 \in S$. But, by the first displayed equation above, $15 \in S$ if and only if $5+3+5(3) = 23 \in S$ and so combining gives $1 \in S$ if and only if $2 \in S$. In a similar manner, combining the third displayed equation above with the fact that $4n+3 = 3+n+3n$ gives that

$$n \in S \iff 3n \in S.$$

So when $n = 1$, we see that $1 \in S \iff 3 \in S$. In particular, since $f(1) = 1$, we see that $1 \in S$ and hence so are both 2 and 3. Now, we prove that $n \in S$ for all $n \in \mathbb{N}$ by strong induction. The base cases have already been proven so we suppose that $n \in S$ for all $1 \le n \le k$ for some $k \in \mathbb{N}$ with $k \ge 3$. We will show that $k+1 \in S$.

If $k+1 \equiv 0 \mod 3$, then $k+1 = 3n$ for some $n \le k$ and by assumption, $n \in S$ and thus, from above, we have that $3n \in S$.

If $k+1 \equiv 1 \mod 3$, then $2k+3 = 2(k+1)+1 \equiv 0 \mod 3$. Then $2k+3 = 3n$ for some integer n. Since $k \ge 3$, we see that $n \le k$ and so as before, $2k+3 \in S$. Finally, since $2(k+1)+1 \in S$, the second displayed equation gives us that $k+1 \in S$.

If $k+1 \equiv 2 \mod 3$, then $k+2 = 2n$ for some $n \in \mathbb{N}$ with $n \ge 2$ since $k \ge 3$. Now, since $2(n-1) \le 3(n-1) = k-1 \le k$, we know that $2(n-1) \in S$ and so $2+(n-1)+2(n-1) = 3n-1 = k+1 \in S$.

Thus, by strong induction, we see that $S = \mathbb{N}$. In particular, $f(2015) = 2015$.