FOCUS ON...

No. 28

Michel Bataille

Some relations in the triangle (II)

Introduction

In this number, we continue our selection of relations in the triangle, focusing on formulas involving lengths related to the classical cevians. As in part I, the notations are standard and borrowed from **[2]**.

About the altitudes h_a, h_b, h_c

Besides the easy

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$$

(which readily follows from $\frac{1}{h_a} = \frac{a}{2F} = \frac{a}{2rs}$ and similar relations), we consider a less known, easy-to-remember formula :

$$(h_a + h_b + h_c)\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) = (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \tag{1}$$

(again an obvious consequence of $2F = ah_a = bh_b = ch_c$).

With the help of the identities

$$(x+y+z)(xy+yz+zx) - 3xyz = \sum_{\text{cyclic}} x^2(y+z) = (x+y)(y+z)(z+x) - 2xyz, \tag{2}$$

we may equivalently write (1) as

$$\frac{(h_a+h_b)(h_b+h_c)(h_c+h_a)}{h_a h_b h_c} = \frac{(a+b)(b+c)(c+a)}{abc} \tag{3}$$

and give a solution to problem **3453** [2009 : 325,328 ; 2010 : 342] that asked for the inequality

$$8\left(\sum_{\text{cyclic}} h_a^2(h_b+h_c)\right) + 16h_a h_b h_c \leq 3\sqrt{3}\left(\sum_{\text{cyclic}} a^2(b+c)\right) + 6\sqrt{3}abc.$$

Indeed, a consequence of (2) is that this inequality is equivalent to $Q \leq \frac{3\sqrt{3}}{8}$ where

$$Q = \frac{(h_a+h_b)(h_b+h_c)(h_c+h_a)}{(a+b)(b+c)(c+a)}.$$

But from (3) we have

$$Q = \frac{h_a h_b h_c}{abc} = \frac{8F^3}{(abc)^2} = \frac{abc}{8R^3} = \sin A \sin B \sin C,$$

hence, using AM-GM and the concavity of the Sine function on $(0,\pi)$,

$$Q \le \left(\frac{\sin A+\sin B+\sin C}{3}\right)^3 \le \left(\sin\left(\frac{A+B+C}{3}\right)\right)^3 = \frac{3\sqrt{3}}{8}.$$

About the distances IA, IB, IC

Prompted by the intervention of IA in part I, we now present a couple of interesting relations connecting the distances IA, IB, IC to other elements of the triangle.

First we consider the product $IA\cdot IB\cdot IC$ and show that

$$sIA\cdot IB\cdot IC = r\cdot abc. \tag{4}$$

The proof is easy :

$$IA\cdot IB\cdot IC = \frac{r}{\sin\frac{A}{2}}\cdot\frac{r}{\sin\frac{B}{2}}\cdot\frac{r}{\sin\frac{C}{2}} = \frac{r^3}{\frac{r}{4R}} = 4Rr^2 = \frac{r}{s}\cdot abc.$$

At this point, it is worth mentioning a beautiful formula that also involves the excenters I_a, I_b, I_c :

$$IA\cdot IB\cdot IC\cdot I_aA\cdot I_bB\cdot I_cC = (abc)^2. \tag{5}$$

To see this, we first remark that

$$bc\cos^2\frac{A}{2} = \frac{bc(1+\cos A)}{2} = \frac{bc}{2}\left(1+\frac{b^2+c^2-a^2}{2bc}\right) = s(s-a), \tag{6}$$

from which we deduce that

$$IA\cdot I_aA = \frac{s-a}{\cos\frac{A}{2}}\cdot\frac{s}{\cos\frac{A}{2}} = bc.$$

Similarly, $IB\cdot I_bB = ca,\ IC\cdot I_cC = ab$ and (5) follows.

Relation (5) reminds us of the known relation $w_aw_bw_cW_aW_bW_c = (abc)^2$ where W_a, W_b, W_c denote the lengths of the angle bisectors extended until they are chords of the circumcircle (see problem **168** [1976 : 136 ; 1977 : 233]). It is interesting to notice that we even have $w_aW_a = bc = IA\cdot I_aA$ and similar relations.

We conclude this paragraph with the formula

$$aIA^2+bIB^2+cIC^2 = abc, \tag{7}$$

from which we will derive a general inequality.

Using (6), we obtain

$$\begin{aligned} aIA^2+bIB^2+cIC^2 &= a\left(\frac{s-a}{\cos\frac{A}{2}}\right)^2 + b\left(\frac{s-b}{\cos\frac{B}{2}}\right)^2 + c\left(\frac{s-c}{\cos\frac{C}{2}}\right)^2 \\ &= a(s-a)^2\frac{bc}{s(s-a)} + b(s-b)^2\frac{ca}{s(s-b)} + c(s-c)^2\frac{ab}{s(s-c)} \\ &= \frac{abc}{s}(s-a+s-b+s-c) \end{aligned}$$

and (7) follows.

A nice application is the inequality

$$aIA \cdot PA + bIB \cdot PB + cIC \cdot PC \geq abc$$

that holds for any point P in the plane of the triangle ABC. To prove it, we use the dot product and the Cauchy-Schwarz inequality as follows :

$$\begin{aligned} aPA \cdot IA + bPB \cdot IB + cPC \cdot IC \\ &= \|a\overrightarrow{IA}\| \|\overrightarrow{IA} - \overrightarrow{IP}\| + \|b\overrightarrow{IB}\| \|\overrightarrow{IB} - \overrightarrow{IP}\| + \|c\overrightarrow{IC}\| \|\overrightarrow{IC} - \overrightarrow{IP}\| \\ &\geq a\overrightarrow{IA} \cdot (\overrightarrow{IA} - \overrightarrow{IP}) + b\overrightarrow{IB} \cdot (\overrightarrow{IB} - \overrightarrow{IP}) + c\overrightarrow{IC} \cdot (\overrightarrow{IC} - \overrightarrow{IP}) \\ &= aIA^2 + bIB^2 + cIC^2 - \overrightarrow{IP} \cdot (a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC}) \\ &= abc \end{aligned}$$

(since $a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC} = \overrightarrow{0}$).

About the exradii r_a, r_b, r_c

Faced with the proof of a relation between the exradii r_a, r_b, r_c, the first move is often to use the equalities

$$F = r_a(s-a) = r_b(s-b) = r_c(s-c). \tag{8}$$

For examples, the striking formulas

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}, \qquad r_a r_b + r_b r_c + r_c r_a = s^2 = \frac{r_a r_b r_c}{r}$$

and

$$\sqrt{\frac{r r_b r_c}{r_a}} + \sqrt{\frac{r r_c r_a}{r_b}} + \sqrt{\frac{r r_a r_b}{r_c}} = s$$

are straightforwardly deduced from (8) and $F = rs = \sqrt{s(s-a)(s-b)(s-c)}$.

With the additional known formulas

$$ab + bc + ca = s^2 + r^2 + 4rR \quad \text{and} \quad a^2 + b^2 + c^2 = 2s^2 - 2r^2 - 8rR,$$

we easily obtain

$$r_a + r_b + r_c = r + 4R \quad \text{and} \quad r^2 + r_a^2 + r_b^2 + r_c^2 + a^2 + b^2 + c^2 = 16R^2$$

that were at work in problem **3570** [2010 : 397,399 ; 2011 : 402].

Since $2F = ah_a = bh_b = ch_c$, one can expect some connections with h_a, h_b, h_c. A good example is

$$\frac{h_b + h_c}{r_a} + \frac{h_b + h_c}{r_a} + \frac{h_b + h_c}{r_a} = 6 \tag{9}$$

which is mentioned but not proved in **[1]**. Here is a quick proof. Since

$$h_b + h_c = 2F\left(\frac{1}{b}+\frac{1}{c}\right) = \frac{2F(ab+ac)}{abc},$$

the left-hand side of (9) rewrites as

$$\frac{2}{abc}\left((ab+ac)(s-a)+(bc+ba)(s-b)+(ca+cb)(s-c)\right) = \frac{2}{abc}\cdot(ab(c)+bc(a)+ca(b))$$

and (9) follows.

The reader will find other formulas of the same kind in exercise **1**.

A mixed formula

A long time ago, I came across the following impressive formula in an old copy of the 1886 *Journal of mathématiques élémentaires Vuibert*,

$$\frac{w_a^2}{h_a}\cdot\sqrt{\frac{m_a^2-h_a^2}{w_a^2-h_a^2}} = 2R. \tag{10}$$

(Of course, a similar result holds if the subscript a is replaced by b or c.). This formula was given with a typo (r instead of R) and without proof!

A possible proof is as follows. With the help of the known formulas

$$w_a^2 = \frac{bc(a+b+c)(b+c-a)}{(b+c)^2}, \qquad h_a = \frac{2F}{a} = \frac{bc}{2R}, \qquad 4m_a^2 = 2b^2+2c^2-a^2$$

and

$$16F^2 = 2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4) = (a+b+c)(b+c-a)(c+a-b)(a+b-c),$$

we first obtain

$$\begin{aligned} w_a^4(4m_a^2-4h_a^2) &= \frac{b^2c^2}{(b+c)^4}\,(a+b+c)^2(b+c-a)^2\left(2b^2+2c^2-a^2-\frac{16F^2}{a^2}\right) \\ &= \frac{b^2c^2(b-c)^2(a+b+c)^2(b+c-a)^2}{a^2(b+c)^2} \end{aligned}$$

and, second,

$$\begin{aligned} &16R^2h_a^2(w_a^2-h_a^2) \\ &= 4b^2c^2\left(\frac{bc(a+b+c)(b+c-a)}{(b+c)^2}-\frac{16F^2}{4a^2}\right) \\ &= \frac{b^2c^2(a+b+c)(b+c-a)}{a^2(b+c)^2}\,(4a^2bc-(b+c)^2(c+a-b)(a+b-c)). \end{aligned}$$

Then (10) follows from $4a^2bc-(b+c)^2(c+a-b)(a+b-c) = (b-c)^2(a+b+c)(b+c-a)$ (as it is readily checked). (Variants of proofs can be found in **[3]**.)

A new youth was recently granted to this relation through several problems composed by Panagiote Ligouras. A typical example is problem **1847** posed in *Mathematics Magazine* in June 2010. Here is the slightly arranged statement :

Prove that in a scalene triangle the following inequality holds

$$\frac{w_a^4(m_a^2-h_a^2)}{h_a^3r_a(w_a^2-h_a^2)}+\frac{w_b^4(m_b^2-h_b^2)}{h_b^3r_b(w_b^2-h_b^2)}+\frac{w_c^4(m_c^2-h_c^2)}{h_c^3r_c(w_c^2-h_c^2)}>\frac{16}{3}.$$

Formula (10) allows a quick proof by immediately transforming the required inequality into

$$\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}>\frac{4r(a+b+c)}{3R^2}. \qquad (11)$$

From exercise **1** below, we deduce

$$\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}=\frac{4(4R+r)}{a+b+c}$$

so that (11) is equivalent to $3R^2(4R+r)>r(a+b+c)^2$. The proof is easily completed by recalling that $R>2r$ and $a+b+c<3\sqrt{3}R$.

Exercises

1. Prove the formulas

$$\left(\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\right)\left(\frac{a+b+c}{r_a+r_b+r_c}\right)=4$$

and

$$\frac{1}{rr_br_c}+\frac{1}{rr_cr_a}+\frac{1}{rr_ar_b}=\frac{8}{h_ah_bh_c}+\frac{1}{r_ar_br_c}.$$

2. (from *College Math. Journal* Problem 937) Prove that in a scalene triangle, the following inequality holds

$$\frac{w_a^4(m_a^2-h_a^2)}{h_a^2(w_a-h_a)\sqrt{w_a\cdot h_a}}+\frac{w_b^4(m_b^2-h_b^2)}{h_b^2(w_b-h_b)\sqrt{w_b\cdot h_b}}+\frac{w_c^4(m_c^2-h_c^2)}{h_c^2(w_c-h_c)\sqrt{w_c\cdot h_c}}\geq 24R^2.$$

References

[**1**] T. Lalesco, *La géométrie du triangle*, J. Gabay, 2003, p. 101-120

[**2**] O. Bottema *et al.*, *Geometric Inequalities*, Wolters-Noordhoff, 1968, p. 9-10

[**3**] Solution to J136, *Mathematical Reflections*, **5**, (2009), p. 10