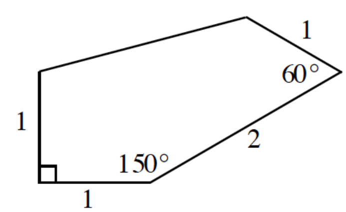
CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(9), p. 373–375.

CC241. Over many centuries, tilings have fascinated mathematicians and the society in general. Particularly interesting are tilings of the plane that use a single type of tile. You can tile the plane with some regular polygons (such as equilateral triangles, squares, regular hexagons). On the other hand, you cannot tile the plane using regular pentagons. Now, we know that some non-regular pentagons can be used to tile the plane, although not all of them are yet known. It was therefore with great enthusiasm that in August 2015, the world welcomed the discovery of a new pentagonal tiling, illustrated below.

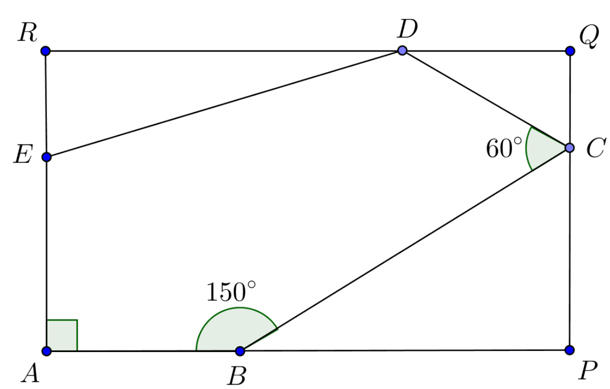


Use the lengths given and angle sizes to calculate the exact area of this pentagon.

Originally Problem 4 from the Scottish Mathematical Council Mathematical Challenge 2016–2017.

We received nine correct solutions and one incorrect submission. We present the solution by Kathleen E. Lewis, slightly expanded by the editor.

We inscribe the pentagon in a rectangle $APQR$ as shown in the diagram, with A set at the origin. Then the area of the pentagon is the area of the rectangle minus the area of the three corner triangles.



By definition of the pentagon, $\angle CBP = 30^\circ$. Since $BC = 2$, we get $CP = 1$, $BP = \sqrt{3}$, and thus $AP = 1 + \sqrt{3}$. Furthermore

$$\angle DCQ = 180^\circ - \angle BCD - \angle PCB = 60^\circ,$$

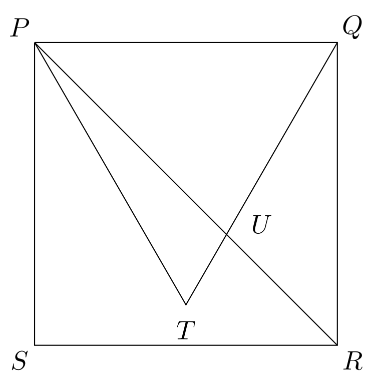
implying $CQ = \frac{1}{2}$ (thus $PQ = \frac{3}{2}$) and $DQ = \frac{\sqrt{3}}{2}$. Since $APQR$ is a rectangle we obtain

$$DR = 1 + \frac{\sqrt{3}}{2} \quad \text{and} \quad ER = \frac{1}{2}.$$

The area of the pentagon is therefore

$$\begin{aligned}[ABCDE] &= AP \cdot PQ - \frac{1}{2}(BP \cdot CP + CQ \cdot DQ + DR \cdot ER) \\ &= \frac{3(1+\sqrt{3})}{2} - \frac{1}{2}\left(\sqrt{3} + \frac{\sqrt{3}}{4} + \frac{2+\sqrt{3}}{4}\right) \\ &= \frac{5+3\sqrt{3}}{4}.\end{aligned}$$

CC242. The diagram below shows square $PQRS$ with sides of length 1 unit. Triangle PQT is equilateral. Show that the area of triangle UQR is $(\sqrt{3}-1)/4$ square units.



Originally Problem 5 from the Scottish Mathematical Council Mathematical Challenge 2016–2017.

We received twelve correct solutions. We present the solution of John Heuver.

Note that $\angle RQT = 30^\circ$ and $\angle PUQ = 75^\circ$ where $\sin\ 75^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$. Using the Law of Sines in $\triangle PUQ$ we find that $UQ = \sqrt{3} - 1$. Letingt $[UQR]$ denote area $\triangle UQR$, we get

$$[UQR] = \frac{1}{2} \cdot UQ \cdot QR \cdot \sin\ 30^\circ = \frac{1}{2}(\sqrt{3}-1) \cdot 1 \cdot \frac{1}{2} = \frac{1}{4}\,(\sqrt{3}-1).$$

CC243. Eight islands each have one or more air services. An air service consists of flights to and from another island, and no two services link the same pair of islands. There are 17 air services in all between the islands. Show that it must be possible to use these air services to fly between any pair of islands.

Originally Problem 2 from the Scottish Mathematical Council Mathematical Challenge 2016–2017.

We received six correct solutions. We present the solution of Joel Schlosberg.

Select an arbitrary island I. Let A be the set of a islands it is possible to reach from I via the network of air services (including both those that require multiple flights and I itself). Since there is at least one air service from I to another island, $a \geq 2$.

Suppose that $a < 8$. Then there is at least one island $J \notin A$. At least one air service connects J with another island K. If $K \in A$, there is a way to fly from I to K and then to J, contradicting the assumption that $J \notin A$. Therefore $J, K \notin A$, so $a \leq 6$. Since $2 \leq a \leq 6$, $|a-4| \leq 2$.

If $J \in A$, $K \notin A$, then any air service connecting J and K would make it possible to fly from I to J to K, contradicting $K \notin A$. Therefore, no air service can connect an island in A to an island not in A. Since each pair of islands has at most one air service between them, at most $\binom{a}{2}$ air services connect the a islands in A, and at most $\binom{8-a}{2}$ air services connect the $8-a$ islands not in A. Then the total number of air services is at most

$$\binom{a}{2} + \binom{8-a}{2} = a^2 - 8a + 28 = 12 + (a-4)^2 \leq 12 + 2^2 = 16,$$

contradicting the assumption that there are 17 air services. Therefore, $a = 8$. That is, all 8 islands are connected to I by some sequence of flights. Then for any pair of islands J, K, it is possible to fly from J to I to K.

CC244. How many distinct solutions consisting of positive integers does this system of equations have?

$$\begin{aligned} x_1 + x_2 + x_3 &= 5, \\ y_1 + y_2 + y_3 &= 5, \\ z_1 + z_2 + z_3 &= 5, \\ x_1 + y_1 + z_1 &= 5, \\ x_2 + y_2 + z_2 &= 5, \\ x_3 + y_3 + z_3 &= 5. \end{aligned}$$

Originally Problem 3 from the Scottish Mathematical Council Mathematical Challenge 2016–2017.

We received 7 submissions of which 3 were correct and complete. We present the solution by Ivko Dimitrić.

We think of a solution of this system as a set of nine numbers (assigned to the variables) placed in nine cells of a 3×3 grid (a table, or a matrix) so that each row sum and each column sum equals 5. Clearly, since all integers are positive, no number greater than or equal to 4 can be considered since the other two numbers in the same row are at least one, producing a row sum of at least 6. Therefore, the numbers that can be used are 1, 2, and 3. There are only two ways (up to permutation) to represent 5 as the sum of three positive integers and they are $5 = 3 + 1 + 1 = 2 + 2 + 1$. The number of 3s that can be placed in the grid can be only 0, 1, or 3. Indeed, the number of 3s used cannot be exactly two, for otherwise the row without 3 is filled with two 2s and one 1, whereas one of the 2s would be in the same column with one of the 3s, making that column sum at least 6.

x_1	x_2	x_3
y_1	y_2	y_3
z_1	z_2	z_3

If the number of 3s used is exactly three, the other six cells must be filled with 1s, and, as long as 3s are placed so that no two are in the same row or the same column, the other cells are filled with 1s to produce a solution to the system. Thus, in this case the number of solutions equals the number of different ways to arrange three 3s in a 3×3 grid so that in any given row or column there is only one 3. That can be done in $3! = 6$ ways since we can place the first 3 in the first row in any of the three cells of the first row, and when that has been done there are two choices to place the next number 3 in the second row and after that the cell for the last 3 in the third row is uniquely determined by the previous choices, which is the cell not in the columns containing the first two 3s chosen.

If the number of 3s is zero (no 3 used in the grid) then the grid is filled with 2s and 1s so that of the three 1s used no two are in the same row or column. The number of ways to arrange three 1s that way is the same as the number of ways to arrange three 3s previously discussed, so it equals $3! = 6$ ways.

There remains the case of having exactly one 3 in the grid. In the column and the row containing that 3 the other four entries are 1s, the remaining four cells in the grid to be filled with 2s. That sole number 3 can be placed in any of the nine cells and when that has been done, use four 1s for the four cells in the same row and the same column where the 3 is and four 2s for the remaining four cells of the grid. So, the cells where 1s and 2s go are completely determined by the choice of the cell for the only 3. Thus there are nine ways in this case.

Altogether, the number of different ways to fill the grid with a combination of 1s, 2s, and 3s so that each row and column sum is 5 equals $6 + 6 + 9 = 21$, so the number of solutions to the system is also 21.

CC245. A pyramid stands on horizontal ground. Its base is an equilateral triangle with sides of length a, the other three edges of the pyramid are of length b and its volume is V. Show that

$$V = \frac{1}{12}a^2\sqrt{3b^2 - a^2}.$$

The pyramid is then placed so that a non-equilateral face lies on the ground. Find the height of the pyramid in this position.

Originally Problem 1 from the Scottish Mathematical Council Mathematical Challenge 2016–2017.

We received seven submissions, out of which six were correct and complete. We present the solution by Joel Schlosberg.

The volume of a pyramid with base B is

$$V = \frac{1}{3}A_B h_B, \tag{1}$$

where A_B is the area of the base and h_B the height with respect to B. First we use (1) with the base B_1 that is an equilateral triangle. Then $A_{B_1} = \sqrt{3}a^2/4$. The apex point of the pyramid, the center of the equilateral triangle (which by symmetry is also the foot of the altitude through the apex point), and an arbitrary vertex of the equilateral triangle form a right triangle with hypotenuse length b and legs of lengths h_{B_1} and $a/\sqrt{3}$ (the distance between the center and any vertex of an equilateral triangle of length a). By the Pythagorean theorem,

$$h_{B_1} = \sqrt{b^2 - \frac{a^2}{3}}.$$

By (1) we obtain

$$V = \frac{1}{3}A_{B_1}h_{B_1} = \frac{1}{3}\cdot\frac{\sqrt{3}a^2}{4}\cdot\sqrt{b^2 - \frac{a^2}{3}} = \frac{1}{12}a^2\sqrt{3b^2 - a^2}. \tag{2}$$

Now let B_2 be one of the bases with sidelengths a, b, and b. The height of this triangle over the side with sidelength a is $\sqrt{b^2 - (a/2)^2}$ and its area thus

$$A_{B_2} = \frac{a}{2}\cdot\sqrt{b^2 - \frac{a^2}{4}} = \frac{1}{4}a\sqrt{4b^2 - a^2}. \tag{3}$$

Using (1), (2), and (3), we obtain

$$\frac{1}{12}a^2\sqrt{3b^2 - a^2} = V = \frac{1}{3}A_{B_2}h_{B_2} = \frac{1}{12}a\sqrt{4b^2 - a^2}\cdot h_{B_2},$$

so the height of the pyramid from a non-equilateral face is

$$a\sqrt{\frac{3b^2 - a^2}{4b^2 - a^2}}.$$