Quadratic Allemands

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1 Sequences with complimentary properties

The sequence of natural numbers, $x_n = n$, has two complementary familiar properties :

$$x_{n+1} + x_{n-1} = 2x_n, \qquad \text{and} \qquad x_{n-1}x_{n+1} = x_n^2 - 1.$$

These two facts can be wrapped up in the fact that, for each n, x_{n-1} and x_{n+1} are the solutions of the quadratic equation

$$h(x, x_n) = 0,$$

where

$$h(x, y) = x^2 - 2xy + (y^2 - 1).$$

In particular, note that this bivariate polynomial is quadratic and *symmetric*. One consequence of this is that, for each n, $h(x_{n-1}, x_n) = h(x_n, x_{n+1})$ so that all the points (x_n, x_{n+1}) lie on the curve $h(x, y) = 0$ in the plane.

The sequence of natural numbers is not the only sequence that exhibits these complementary properties involving the sum and product of the terms adjacent to a given term. As an exercise, discover analogous equations for each of the sequences

$$\{0, 1, 3, 8, 21, 55, 144, \ldots\}$$

and

$$\{1, 2, 5, 13, 34, 89, 233, \ldots\}$$

formed by taking alternate terms of the Fibonacci sequence. For each of them, determine a symmetric quadratic polynomial $h(x, y)$ such that x_{n-1} and x_{n+1} are the solutions of the polynomial equation $h(x, x_n) = 0$. Identify the curves that contain the two sets of points

$$(0, 1), (1, 3), (3, 8), (8, 21), (21, 55), (55, 144), \ldots$$

and

$$(1, 2), (2, 5), (5, 13), (13, 34), (34, 89), (89, 233), \ldots.$$

2 Quadratic allemands

We are led to generalize the situation as follows. Start with the general symmetric quadratic polynomial

$$h(x, y) = \alpha(x^2 + y^2) + \beta xy + \gamma(x + y) + \delta.$$

Since our interest will be in the solutions of the equation $h(x, y) = 0$, we will assume that $\alpha = 1$. We define a *quadratic allemand* as a bilateral sequence

$$\{x_n : n = 0, \pm 1, \pm 2, \pm 3, \ldots\}$$

for which a *seed* x_0 is given. x_{-1} and x_1 are defined to be the two solutions of the quadratic equation $h(x, x_0) = 0$.

We can define the remaining terms recursively going in both directions from x_0. Suppose that we have determined $x_0, x_1, \ldots, x_m$ for $m > 0$, so that

$$0 = h(x_m, x_{m-1}) = h(x_{m-1}, x_m).$$

Then x_{m-1} is one solution of $h(x, x_m) = 0$ and we define x_{m+1} to be the second solution of this equation. A similar definition can be used for negative indices.

Write

$$h(x, y) = x^2 + y^2 + \beta xy + \gamma(x + y) + \delta = x^2 + (\beta y + \gamma)x + (y^2 + \gamma y + \delta).$$

Any allemand corresponding to this function must satisfy both of the recursions

$$x_{n+1} + x_{n-1} = -(\beta x_n + \gamma) \qquad (1)$$

$$x_{n+1}x_{n-1} = x_n^2 + \gamma x_n + \delta \qquad (2)$$

However, it turns out that any sequence that satisfies either of the recursions (1) and (2) are allemands. If (1) is satisfied, then it is straightforward to show, for each n, that

$$x_{n+1}x_{n-1} - x_n^2 - \gamma x_n = x_n x_{n-2} - x_{n-1}^2 - \gamma x_{n-1}$$

so that $x_{n+1}x_{n-1} - x_n^2 - \gamma x_{n-1}$ is an invariant for any recursion satisfying (1) alone. If we let δ be the value of this invariant, then we have (2) holding as well, so that any sequence (1) turns out to be an allemand.

With a little more trouble, it can be shown that, if (2) holds, then

$$x_{n+1} + x_{n-1} + \gamma = \frac{x_n}{x_{n-1}}(x_n + x_{n-2} + \gamma)$$

so that $x_n^{-1}(x_{n+1} + x_{n-1} + \gamma)$ is an invariant $-\beta$ and

$$x_{n+1} + \beta x_n + x_{n-1} + \gamma = 0.$$

Thus, any sequence defined by (2) alone is in fact an allemand.

We note the relations :

$$h(y, -(\beta y + x + \gamma)) = h(x, y)$$

and

$$h\left(y, \frac{y^2 + \gamma y + \delta}{x}\right) = \frac{y^2 + \gamma y + \delta}{x^2} h(x, y)$$

with the result that $h(x, y)$ is an invariant for two consecutive terms of sequence (1) and $h(x, y)/xy$ is an invariant for two consecutive terms of (2).

3 Special cases

There are a number of special cases worth investigating. In each case, determine the recursions (1) and (2), when there is an allemand consisting completely of real terms, when the sequence is periodic, and the curve that contains all points (x_n, x_{n+1}).

(1) $\beta = -2$.

(2) $\beta = 0$.

(3) $\beta = 2$. Consider the cases that $\gamma^2 - \delta$ is positive, negative and zero.

(4) $(\beta, \gamma, \delta) = (2, -3, 2)$.

(5) $\gamma = \delta = 0$.

(6) $\gamma = 0, \delta = -1$.

4 Application

One application of this theory is determining when a sequence that satisfies a recursion relation (2) has all integer entries. Consider for example the sequence defined by

$$x_0 = 1, \qquad x_1 = -1, \qquad \text{and} \qquad x_{n+1} = (x_n^2 - x_n + 1)/x_{n-1}$$

for $n \geq 1$. The first few terms of this sequence are

$$1, -1, 3, -7, 19, -49, 129, -337, 883, -2311.$$

Show that this is the positive part of a quadratic allemand with seed 1 and thus prove that each of its entries is an integer.

5 Cubic and quartic allemands

We can also talk about cubic and quartic allemands, where the function $h(x, y)$ is symmetric and respectively cubic and quartic while being quadratic in each of its variables. In the cubic case, we have

$$\begin{aligned} h(x, y) &= x^2y + xy^2 + \alpha(x^2 + y^2) + \beta xy + \gamma(x + y) + \delta \\ &= (y + \alpha)x^2 + (y^2 + \beta y + \gamma)x + (\alpha y^2 + \gamma y + \delta). \end{aligned}$$

and, in the quartic case,

$$\begin{aligned} h(x, y) &= x^2y^2 + \alpha xy(x + y) + \beta(x^2 + y^2) + \gamma xy + \delta(x + y) + \epsilon \\ &= (y^2 + \alpha y + \beta)x^2 + (\alpha y^2 + \gamma y + \delta)x + (\beta y^2 + \delta y + \epsilon). \end{aligned}$$

As for quadratic allemands, the allemands produced by these functions have a related pair of recursion relations corresponding to (1) and (2). But this is a story for another day.

6 Appendix

We provide some brief notes on the questions of the foregoing sections. The sequence $\{0, 1, 3, 8, 21, \ldots\}$ corresponds to the quadratic $h(x, y) = x^2 - 3xy + y^2 - 1$ and the sequence $\{1, 2, 5, 13, 34, \ldots\}$ corresponds to $h(x, y) = x^2 - 3xy + y^2 + 1$.

Here are the computations in full relating (1) and (2) in Section 2.

$$\begin{aligned} x_{n+1}x_{n-1} - x_n^2 - \gamma x_n &= -x_{n-1}(\beta x_n + \gamma + x_{n-1}) - x_n^2 - \gamma x_n \\ &= -x_n(\beta x_{n-1} + x_n + \gamma) - \gamma x_{n-1} - x_{n-1}^2 \\ &= x_n x_{n-2} - x_{n-1}^2 - \gamma x_{n-1}; \end{aligned}$$

$$\begin{aligned} x_{n+1} + x_{n-1} + \gamma &= \frac{x_n^2 + \gamma x_n + \delta}{x_{n-1}} + x_{n-1} + \gamma \\ &= \frac{1}{x_{n-1}}(x_n^2 + \gamma x_n + \delta + x_{n-1}^2 + \gamma x_{n-1}) \\ &= \frac{x_n}{x_{n-1}}\left(x_n + \gamma + \frac{x_{n-1}^2 + \gamma x_{n-1} + \delta}{x_n}\right) \\ &= \frac{x_n}{x_{n-1}}(x_n + x_{n-2} + \gamma). \end{aligned}$$

For some special cases in Section 3, we have the following :

(1) The allemand is an arithmetic progression with common difference d given by $d^2 = -\delta$.

(2) The allemand has period 4 and the points (x_n, x_{n+1}) lie on a circle with centre $(-\frac{1}{2}, -\frac{1}{2})$ and radius $\sqrt{\frac{1}{2}\gamma^2 - \delta}$.

(5) This is a geometric progression with common ratio r satisfying $r^2 + \beta r + 1$.

In Section 4, the sequence satisfies the recursion

$$x_{n+1} + x_{n-1} = -3x_n + 1$$

and the allemand is produced by

$$h(x, y) = x^2 + y^2 + 3xy - (x + y) + 1.$$