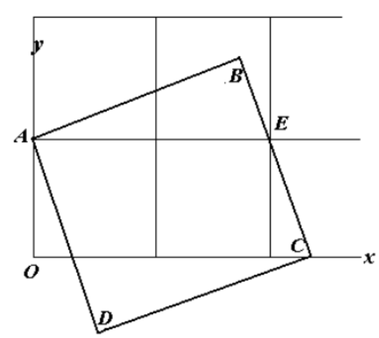
SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2016: 42(7), p. 313–317.

4161. *Proposed by Peter Y. Woo.*

A high-school math teacher discovered some geometry problems while sliding a rug under his feet, over a floor with square tiles of length 1 unit. He chose x and y axes along two edges of some arbitrary tile. Today, he moved the square rug $ABCD$ of length between 1 and 2 units, so that A is on $(0, 1)$ and C is on $(c, 0)$ for some $c > 2$. Then surprise! He noticed that the edge BC goes through the point $(2, 1)$. Can you find $\angle BAE$?



We received 34 *submissions, out of which* 33 *were complete and correct. We present two of the most elegant solutions.*

Solution 1, by Kee-Wai Lau, slightly extended by the editor.

Let F be the point $(2, 0)$ and let $\theta = \angle BAE = \angle FEC$. Then $AB = 2\cos\theta$, $BE = 2\sin\theta$ and

$$EC = BC - BE = AB - BE = 2(\cos\theta - \sin\theta).$$

Thus

$$1 = EF = EC\cos\theta = 2\cos\theta(\cos\theta - \sin\theta),$$

or

$$\cos(2\theta) - \sin(2\theta) = 0,$$

which implies

$$\tan(2\theta) = 1$$

and hence $\theta = \frac{\pi}{8}$.

Solution 2, by Ivko Dimitrić, slightly modified by the editor.

Let the vertical line $x = 1$ intersect the lines AE and AB at points P and Q respectively, and let F be the point $(2, 0)$. Then the right triangles APQ and EFC are congruent, having two congruent angles $\angle QAP = \angle CEF$ and adjacent legs each of unit length. Hence $AQ = EC$, and thus $BQ = BE$, which implies that QEB is a right isosceles triangle. Therefore

$$\angle AQE = \pi - \angle BQE = \frac{3\pi}{4}.$$

Since PQ halves AE at a right angle, the triangle AQE is also isosceles, and hence

$$\angle BAE = \angle QAE = \frac{\pi}{8}.$$

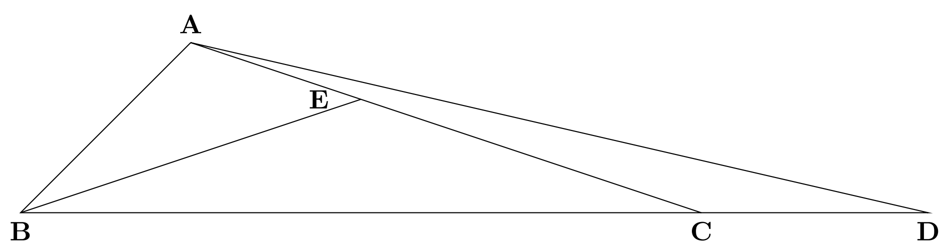
4162. *Proposed by George Apostolopoulos.*

Let ABC be a triangle such that $\angle B = 2\angle C$. We extend the side BC by a segment CD equal to $\frac{1}{3}BC$. Prove that

$$\text{Area}(ABC) = \frac{1}{4}|BC|^2 \cdot \cot\frac{\theta}{2},$$

where $\theta = \angle BAD$.

There were 19 correct solutions. We present five different solutions. The first two solutions are adapted from those of Daniel Dan, George Florin Serban, Titu Zvonaru and the proposer.



Solution 1.

Let a, b, c, d be the respective lengths of BC, CA, AB, AD, and suppose that the bisector of angle ABC intersects AC at E. Then, from the similarity of triangles AEB and ABC and the equality $BE = EC = ab(a+c)^{-1}$, we have that $AC : AB = BC : BE$, whence

$$b^2 = c(a + c). \tag{1}$$

Applying Stewart's Theorem to triangle ABD and cevian AC (or alternatively the Cosine Law to triangles ABC and ACD and eliminating the cosine of the angle at C), we find that

$$ad^2 + \left(\frac{a}{3}\right)c^2 = \left(b^2 + \frac{a^2}{3}\right)\left(\frac{4a}{3}\right).$$

This, along with (1), leads to $9d^2 = 4a^2 + 12b^2 - 3c^2 = (3c+2a)^2$, whence

$$d = \frac{1}{3}(3c+2a). \tag{2}$$

Applying the Cosine Law to triangle ABD and using (2) yields that

$$16a^2 = 9(c^2 + d^2 - 2cd\cos\theta) = (18c^2 + 12ac + 4a^2) - 6c(3c+2a)\cos\theta,$$

so that

$$6(3c^2 + 2ac)\cos\theta = 6(3c^2 + 2ac - 2a^2),$$

whence

$$\cos\theta = 1 - \frac{2a^2}{3c^2 + 2ac} = 1 - \frac{2a^2}{3cd}, \tag{3}$$

and

$$cd = \frac{a^2}{3\sin^2\frac{\theta}{2}}.$$

Therefore

$$[ABC] = \frac{3}{4}[ABD] = \frac{3}{8}cd\sin\theta = \frac{1}{4}a^2\cot^2\frac{\theta}{2}.$$

Solution 2.

We begin by deriving an expression for $\cot\theta/2$. As in Solution 1, we can use (1) and (2) to derive (3), from which

$$1 - \cos\theta = \frac{2a^2}{3c^2 + 2ac}, \qquad 1 + \cos\theta = \frac{6c^2 + 4ac - 2a^2}{3c^3 + 2ac},$$

and

$$\cot^2\frac{\theta}{2} = \frac{1+\cos\theta}{1-\cos\theta} = \frac{(c+a)(3c-a)}{a^2}. \tag{4}$$

From Heron's area formula, (1) and (4), we obtain that

$$\begin{aligned} 16[ABC]^2 &= (a+c+b)(a+c-b)(b-a+c)(b+a-c) \\ &= [(a+c)^2 - b^2][b^2 - (c-a)^2] = [a(a+c)][a(3c-a)] \\ &= a^4\cot^2\frac{\theta}{2}. \end{aligned}$$

whence $[ABC] = \frac{a^2}{4}\cot\frac{\theta}{2}$ as desired.

Solution 3, by C.R. Pranesachar.

Using the notation of Solution 1 and the fact that $3d = 2a + 3c$, we note that the triangle ABD has sides c, d and $4a/3$ and semiperimeter $a + c$. Applying the formula

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{(s-b)(s-c)}{[ABC]}$$

for an arbitrary triangle ABC to the triangle ABD and angle θ, we obtain

$$\cot\frac{\theta}{2} = \frac{[ABD]}{a^2/3} = \frac{4[ABC]}{a^2}$$

as desired.

Solution 4, by Leonard Giugiuc.

Place the triangle in the Cartesian plane with A at $(0,1)$ and C at $(k,0)$ with $k = \cot C > 0$. We find that the coordinates of A, B, C, D are given by

$$A(0,1), \quad B\left(\frac{1-k^2}{2k}, 0\right), \quad C(k,0), \quad D\left(\frac{9k^2-1}{6k}, 0\right),$$

and that $a = (3k^2-1)/(2k) = 2[ABC]$. Since $x = \cot\frac{1}{2}\angle BDA$ is the positive solution of the equation

$$\frac{x^2-1}{2x} = \frac{9k^2-1}{6k},$$

we have $x = 3k$ and $\tan\frac{1}{2}\angle BDA = 1/(3k)$. Observe that $\theta + B + \angle BDA = \pi$ so that

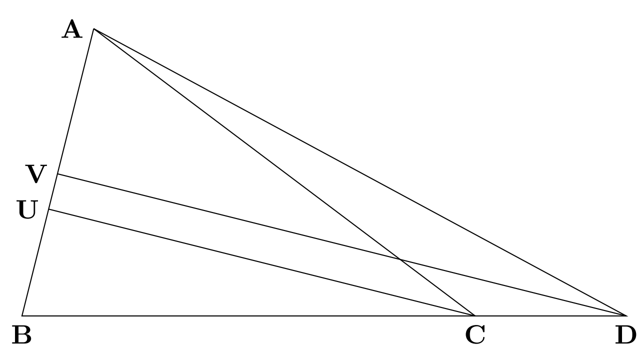
$$\cot\frac{\theta}{2} = \tan\left(C + \frac{\angle BDA}{2}\right) = \frac{4k}{3k^2-1} = \frac{2}{a}.$$

Therefore

$$\frac{1}{4}a^2\cot\frac{\theta}{2} = \frac{a}{2} = [ABC].$$

Solution 5, by Andrea Fanchini.

Let U and V be the respective feet of the perpendiculars from C and D to the side AB. Since $UV : BU = CD : BC = 1 : 3$, then $3(AU - AV) = 3UV = BU$.



(To cater to the various configurations, the lengths along the vector $\overrightarrow{AB}$ are signed, a contingency that can be accommodated using barycentric coordinates.) Since $DV : CU = BD : BC = 4 : 3$, then

$$\cot\theta = \frac{AV}{DV} = \frac{3AV}{4CU} = \frac{1}{4}\left(\frac{3AU - BU}{CU}\right) = \frac{1}{4}(3\cot A - \cot B).$$

Let $S = 2[ABC]$, twice the area of the triangle, and define $S_\phi = S\cot\phi$ for any angle ϕ. In particular

$$S_A = \frac{S\cos A}{\sin A} = bc\cos A = \frac{1}{2}(b^2 + c^2 - a^2),$$

with analogous expressions for S_B and S_C. It follows that

$$S_B + S_C = a^2 \quad \text{and} \quad 4S_\theta = 3S_A - S_B.$$

Since

$$2\cot B\cot C = 2\cot 2C\cot C = \cot^2 C - 1,$$

we have

$$2S_BS_C = S_C^2 - S^2,$$

so that

$$\begin{aligned} a^2S_B = S_B(S_B + S_C) &= (S_B + S_C)^2 + (2S_BS_C - S_C^2) - 3S_BS_C \\ &= a^4 - S^2 - 3S_BS_C = a^4 + 3(S^2 - S_BS_c) - 4S^2. \end{aligned}$$

From the identity $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$, we obtain

$$a^2S_B = a^4 + 3S_A(S_B + S_C) - 4S^2 = a^4 + 3a^2S_A - 4S^2,$$

which can be rearranged to yield

$$\frac{3S_A - S_B}{4} = \frac{4S^2 - a^4}{4a^2}.$$

We turn to the desired equality, which can be recast as

$$S_{\frac{\theta}{2}} = \frac{2S^2}{a^2}.$$

Since, for $0 < \phi < \pi$, there is a one-one relation between $S_\phi = (S_{\phi/2}^2 - S^2)/(2S_\phi/2)$ and $S_{\phi/2}$, the equality is equivalent to

$$\frac{3S_A - S_B}{4} = S_\theta = \frac{(2S^2/a^2)^2 - S^2}{4S^2/a^2} = \frac{4S^2 - a^4}{4a^2},$$

which has already been established.

Editor's Comments. The solutions revealed several interesting relations, in particular expressions for the cotangent in terms of the elements of triangle ABD:

$$\cot\frac{\theta}{2} = \frac{2\sin C\sin 2c}{\sin 3C} = \frac{\sqrt{2b^2+c^2-a^2}}{a}.$$

Václav Konečný pointed out that if angle C is less than $45°$, then drawing the circle with centre A through B along with the radius to its other intersection with BC gives the diagram for the trisection of the exterior angle at A with a marked straightedge and compasses.

Miguel Amengual Covas observed that the condition $\angle B = 2\angle C$ figured in earlier problems published in ***Crux*** and references to those can be found in the article *Recurring **Crux** Configurations: No. 7* by Chris Fisher in Volume 38(6) of June 2012, p. 238–240.

4163. *Proposed by Leonard Giugiuc.*

Let a, b be real numbers with $0 < a < b$ and consider a positive sequence x_n such that

$$\lim_{n\to\infty}\left(ax_n + \frac{b}{x_n}\right) = 2\sqrt{ab}.$$

Find $\lim_{n\to\infty} x_n$ or show that it does not exist.

We received 17 submissions, all of which were correct. We present two different solutions.

Solution 1, by Adnan Ali.

Define the sequence $\{y_n\}$ as $y_n = ax_n + \dfrac{b}{x_n}$. Then $\lim_{n\to\infty} y_n = 2\sqrt{ab}$. Solving

$$ax_n^2 - y_nx_n + b = 0,$$

we have

$$x_n = \frac{1}{2a}\left(y_n \pm \sqrt{y_n^2 - 4ab}\right).$$

Since

$$\lim_{n\to\infty}\sqrt{y_n^2-4ab} = 0,$$

it follows that

$$\lim_{n\to\infty} x_n = \frac{1}{2a}\lim_{n\to\infty} y_n = \frac{2\sqrt{ab}}{2a} = \sqrt{\frac{b}{a}}.$$

Solution 2, by Michel Bataille.

From

$$\left(ax_n + \frac{b}{x_n}\right)^2 - \left(ax_n - \frac{b}{x_n}\right)^2 = 4ab$$

and the hypothesis, we deduce that

$$\lim_{n\to\infty}\left(ax_n - \frac{b}{x_n}\right)^2 = 0, \quad \text{so} \quad \lim_{n\to\infty}\left(ax_n - \frac{b}{x_n}\right) = 0.$$

Since

$$2ax_n = \left(ax_n + \frac{b}{x_n}\right) + \left(ax_n - \frac{b}{x_n}\right),$$

it follows that $\lim_{n\to\infty} 2ax_n = 2\sqrt{ab}$, and so $\lim_{n\to\infty} x_n = \sqrt{\dfrac{b}{a}}$.

Editor's comment. Both Roy Barbara and Oliver Geupel pointed out that the assumption $a < b$ is superfluous.

4164. *Proposed by G. Di Bona, A. Fiorentino, A. Moscariello and G. G. N. Angilella.*

In an election, N voters are to elect k representatives. Each voter must indicate exactly m distinct preferences, with $m \leq k < N$. Every voter is a candidate themselves, and all candidates have a distinct age. The candidates are then ranked according to the number of votes received, and the k candidates who receive the largest number of votes are elected. In case of degeneracies, the eldest candidate is elected.

What is the minimum number of votes that a candidate should receive, in order to be sure to get elected?

There were five submissions, but only that of the proposer was complete and correct. The faulty solutions tended to argue from extreme situations without making a solid connection to the general one, however intuitively appealing this might be. Two solvers gave answers that depended on the status of particular individuals, rather than a value V that applied to any of the candidates. We present the solution of the proposer.

The minimum number of votes that a candidate should receive in order to ensure election is

$$V = \left\lfloor \frac{Nm}{k+1} \right\rfloor + 1.$$

First, we show that a candidate receiving at least V votes will be elected. Let $v_1, v_2, \ldots, v_{k+1}$ be the number of votes received by the top $k+1$ candidates, with $v_1 \geq v_2 \geq \cdots \geq v_{k+1}$. Since $v_1 + v_2 + \cdots + v_{k+1} \leq Nm$, the total number of votes cast, the arithmetic mean M of these numbers must satisfy

$$v_{k+1} \leq M \leq \frac{Nm}{k+1} < V.$$

Hence, any candidate with at least V votes must be among the top k candidates and so be elected.

On the other hand, we construct a situation in which a candidate with $V-1$ votes fails to get elected. Suppose only $k+1$ candidates receive votes, and we number them $1, 2, \ldots, k+1$. Create a sequence with Nm terms by repeating the base sequence $\{1, 2, \ldots, k+1\}$ as many times as needed to fill the Nm slots. Partition this sequence into N subsequences of m terms, and let the ith voter vote for the candidates in the ith subsequence. Each candidate will receive at least $\lfloor \frac{Nm}{k+1} \rfloor = V-1$ votes and the candidate numbered $k+1$ will receive exactly $V-1$ votes. Thus, this candidate may fail to get elected.

Note that the role of the age rule is to break a tie. Consider the situation where the N voters are arranged in a circle and each votes for the m candidates sitting immediately to the right. Then each candidate receives the same number m of votes, and we need to decide on the winners by seniority.

4165. *Proposed by Daniel Sitaru.*

Prove that for all real numbers x_1, x_2, x_3 and x_4, we have,

$$|x_1 + x_2 + x_3 + x_4| + 2(|x_1| + |x_2| + |x_3| + |x_4|) \geq 6 \sqrt[6]{\prod_{1 \leq i < j \leq 4} |x_i + x_j|}.$$

We received six submissions, all of which were correct. We present the solution with generalization by Michel Bataille.

We prove the stronger result that for any complex numbers x_1, x_2, x_3 and x_4, we have

$$|x_1 + x_2 + x_3 + x_4| + 2(|x_1| + |x_2| + |x_3| + |x_4|) \geq \sum_{1 \leq i < j \leq 4} |x_i + x_j|. \qquad (1)$$

The proposed inequality then follows from (1) by the AM-GM Inequality. To prove (1), we will make use of Hlawka's inequality which states that

$$|a+b+c| + |a| + |b| + |c| \geq |a+b| + |b+c| + |c+a| \qquad (2)$$

for all complex numbers a, b, c. (See e.g., problem **2482** [1999 : 430 ; 2000 : 506].) Setting $a = x_1, b = x_2$ and $c = x_3 + x_4$, then from (2) we have

$$|x_1+x_2+x_3+x_4|+|x_1|+|x_2|+|x_3+x_4| \geq |x_1+x_2|+|x_2+x_3+x_4|+|x_1+x_3+x_4|. \qquad (3)$$

Applying (2) again, we obtain

$$|x_2 + x_3 + x_4| \geq |x_2 + x_3| + |x_3 + x_4| + |x_2 + x_4| - |x_2| - |x_3| - |x_4| \qquad (4)$$

and

$$|x_1 + x_3 + x_4| \geq |x_1 + x_3| + |x_3 + x_4| + |x_1 + x_4| - |x_1| - |x_3| - |x_4| \qquad (5)$$

Adding (4) and (5) and denoting the right side of (3) by R, then we have

$$R \geq |x_3 + x_4| - |x_1| - |x_2| - 2|x_3| - 2|x_4| + \sum_{1 \leq i < j \leq 4} |x_i + x_j|. \qquad (6)$$

From (3) and (6), we deduce that

$$\begin{aligned}&|x_1+x_2+x_3+x_4|+|x_1|+|x_2|+|x_3+x_4|\\&\quad\geq |x_3+x_4|-|x_1|-|x_2|-2|x_3|-2|x_4|+\sum_{1\leq i<j\leq 4}|x_i+x_j|\end{aligned}$$

from which (1) follows immediately.

4166. *Proposed by Mihaela Berindeanu.*

Show that for all real numbers x, y and z, we have:

$$2^{4x-y}+2^{4y-z}+2^{4z-x}\geq 2^{x+2y}+2^{y+2z}+2^{z+2x}.$$

We received 14 solutions. We present three solutions.

Solution 1, by Adnan Ali.

Let $2^x=a, 2^y=b$ and $2^z=c$. Then the proposed inequality becomes

$$\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{a}\geq ab^2+bc^2+ca^2,$$

which is nothing but a consequence of the Cauchy-Schwarz Inequality:

$$\left(\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{a}\right)(bc^2+ca^2+ab^2)\geq (a^2c+b^2a+c^2b)^2=(bc^2+ca^2+ab^2)^2.$$

Equality holds if and only if $a=b=c$ or equivalently $x=y=z$.

Solution 2, by Salem Malikić.

Introducing substitution $2x=a$, $2y=b$ and $2z=c$ our inequality becomes equivalent to

$$\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{a}\geq ab^2+bc^2+ca^2.$$

The last one follows after adding up the following obvious inequalities

$$\frac{a^4}{b}+bc^2\geq 2a^2c,\quad \frac{b^4}{c}+ca^2\geq 2b^2a,\quad \frac{c^4}{a}+ab^2\geq 2c^2b.$$

In order to achieve equality one must have

$$\frac{a^4}{b}=bc^2,\quad \frac{b^4}{c}=ca^2,\quad \frac{c^4}{a}=ab^2.$$

That implies $a^2=bc$, $b^2=ca$ and $c^2=ab$, i.e.

$$(a-b)^2+(b-c)^2+(c-a)^2=0,$$

so $a=b=c$, i.e. $x=y=z$.

Solution 3, by Michel Bataille.

Multiplying both sides by the positive real number $2^{-(x+y+z)}$, we obtain

$$2^{3x-2y-z} + 2^{3y-2z-x} + 2^{3z-2x-y} \geq 2^{y-z} + 2^{z-x} + 2^{x-y}$$

or, setting $a = 2^{y-z},\ b = 2^{z-x},\ c = 2^{x-y}$,

$$ac^3 + ba^3 + cb^3 \geq a + b + c. \tag{1}$$

Thus, it suffices to prove (1) for positive reals a, b, c such that $abc = 1$. Since $ac^3 = c(abc)\frac{c}{b} = c \cdot \frac{c}{b}$ and similarly $ba^3 = a \cdot \frac{a}{c},\ cb^3 = b \cdot \frac{b}{a}$, (1) rewrites as $X \geq 1$ where

$$X = \frac{a}{a+b+c} \cdot f\left(\frac{c}{a}\right) + \frac{b}{a+b+c} \cdot f\left(\frac{a}{b}\right) + \frac{c}{a+b+c} \cdot f\left(\frac{b}{c}\right)$$

where the function f is defined by $f(t) = \frac{1}{t}$.

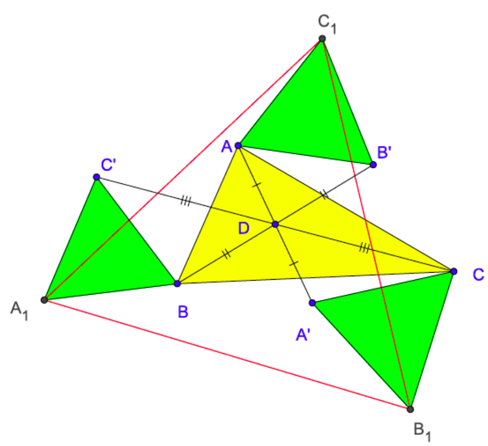
Now, since f is convex on the interval $(0, \infty)$, Jensen's inequality yields

$$X \geq f\left(\frac{a}{a+b+c} \cdot \frac{c}{a} + \frac{b}{a+b+c} \cdot \frac{a}{b} + \frac{c}{a+b+c} \cdot \frac{b}{c}\right) = f(1) = 1$$

and therefore (1) holds.

4167. *Proposed by Dao Thanh Oai and Leonard Giugiuc.*

Consider triangle ABC and let D be any point in the plane. Let points A', B', C' be reflections of points A, B, C in D, respectively. Construct the 3 triangles $AB'C_1$, $CA'B_1$ and $BC'A_1$ outwardly as the given diagram indicates:



Show that $A_1B_1C_1$ is an equilateral triangle.

We received 13 submissions, all correct. We feature two solutions; the first is typical of the ten that used complex coordinates.

Solution 1, by Somasundaram Muralidharan.

We must assume that the triangles $AB'C_1, CA'B_1$, and $AB'C_1$ are equilateral with the same orientation as ΔABC, but there is no loss of generality in assuming that D is at the origin. Let a, b, c be the complex numbers representing the vertices A, B, C respectively. Then A', B', C' are represented by the complex numbers $-a, -b, -c$ respectively. The vertex B_1 is obtained by rotating the segment CA' about vertex C anticlockwise through $60°$. Hence, if the complex number representing B_1 is b_1, then

$$b_1 = c + ((-a) - c)e^{\frac{i\pi}{3}} = c - (a + c)e^{\frac{i\pi}{3}}.$$

Similarly the complex numbers representing C_1 and A_1, namely c_1 and a_1, are given by

$$c_1 = a - (a + b)e^{\frac{i\pi}{3}}, \qquad a_1 = b - (b + c)e^{\frac{i\pi}{3}}.$$

Now,

$$\begin{aligned}\overrightarrow{C_1A_1} &= (b - (b + c)e^{\frac{i\pi}{3}}) - (a - (a + b)e^{\frac{i\pi}{3}}) \\ &= (b - a) + (a - c)e^{\frac{i\pi}{3}} \\ &= b - ce^{\frac{i\pi}{3}} - a(1 - e^{\frac{i\pi}{3}}) \\ &= b - ce^{\frac{i\pi}{3}} - ae^{-\frac{i\pi}{3}}\end{aligned}$$

Similarly,

$$\overrightarrow{C_1B_1} = -a + be^{\frac{i\pi}{3}} + ce^{-\frac{i\pi}{3}}.$$

Because

$$e^{\frac{i2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = -\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -e^{-\frac{i\pi}{3}},$$

we have

$$\overrightarrow{C_1B_1} = \overrightarrow{C_1A_1}e^{\frac{i\pi}{3}}.$$

Thus $\overrightarrow{C_1B_1}$ is obtained by rotating $\overrightarrow{C_1A_1}$ anticlockwise by $60°$ and the triangle $A_1B_1C_1$ is equilateral.

Solution 2, by Peter Y. Woo, with small corrections supplied by the editor.

Because the hexagon $AC'BA'CB'$ is symmetric about D, the quadrilaterals $BC'B'C$ and $ABA'B'$ are parallelograms. Compare the vector $\overrightarrow{B_1A_1}$, which is the sum of vectors $\overrightarrow{B_1A'}, \overrightarrow{A'B}$, and $\overrightarrow{BA_1}$, to the vector $\overrightarrow{B_1C_1}$, which is the sum of vectors $\overrightarrow{B_1C}, \overrightarrow{CB'}$, and $\overrightarrow{B'C_1}$:

$\overrightarrow{B_1A'}$ equals $\overrightarrow{B_1C}$ rotated $60°$ anticlockwise,
$\overrightarrow{A'B}$ equals $\overrightarrow{B'C_1}$ rotated $60°$ anticlockwise,
$\overrightarrow{BA_1}$ equals $\overrightarrow{CB'}$ rotated $60°$ anticlockwise.

Because vector addition is commutative, the sum of the first vector of each of the three pairs equals the sum of the last three vectors rotated 60° anticlockwise. Consequently $\overrightarrow{A_1B_1}$ is the vector $\overrightarrow{C_1B_1}$ rotated 60° anticlockwise. It follows that $A_1B_1C_1$ is an equilateral triangle.

4168. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 18$ and $abc = 4$. Prove that

$$6 \leq a + b + c \leq 2\sqrt{2\sqrt{6} + 4} + \sqrt{6} - 2.$$

When does equality hold?

We received 15 solutions, all of which were correct. We present the one by Joseph DiMuro.

We'll use Lagrange multipliers to find the extreme values of $f(a, b, c) = a + b + c$, subject to the constraints $g_1(a, b, c) = a^2 + b^2 + c^2 = 18$ and $g_2(a, b, c) = abc = 4$.

Note first that the extreme values must occur at points where $\bigtriangledown f = \lambda_1 \bigtriangledown g_1 + \lambda_2 \bigtriangledown g_2$ for some real numbers λ_1, λ_2, unless they occur at points where the vectors $\bigtriangledown g_1$ and $\bigtriangledown g_2$ are linearly dependent.

Since $\bigtriangledown f = (1, 1, 1,])$, $\bigtriangledown g_1 = (2a, 2b, 2c)$, and $\bigtriangledown g_2 = (bc, ca, ab)$, $\bigtriangledown g_1$ and $\bigtriangledown g_2$ are linearly dependent only if $a = b = c$. But the constraints $a^2 + b^2 + c^2 = 18$ and $abc = 4$ cannot be satisfied if $a = b = c$ since the system $3a^2 = 18$ and $a^3 = 4$ has no solutions.

Hence it suffices to look for points that satisfy $\bigtriangledown f = \lambda_1 \bigtriangledown g_1 + \lambda_2 \bigtriangledown g_2$. That is, $\lambda_1(2a, 2b, 2c) + \lambda_2(bc, ca, ab) = (1, 1, 1)$, or

$$2a\lambda_1 + bc\lambda_2 = 1 \quad (1)$$
$$2b\lambda_1 + ac\lambda_2 = 1 \quad (2)$$
$$2c\lambda_1 + ab\lambda_2 = 1 \quad (3)$$

From (1)-(2) we obtain

$$2(a - b)\lambda_1 + c(b - a)\lambda_2 = 0 \iff 2(a - b)\lambda_1 = c(a - b)\lambda_2,$$

which is true if and only if either $a = b$ or $2\lambda_1 = c\lambda_2$. Similarly, from (2)-(3) and (3)-(1) we deduce that either $b = c$ or $2\lambda_1 = a\lambda_2$; and either $c = a$ or $2\lambda_1 = b\lambda_2$.

Recall that $a = b = c$ is impossible. On the other hand, if a, b, and c are all distinct, then $2\lambda_1 = c\lambda_2 = b\lambda_2 = a\lambda_2$ would yield $\lambda_1 = \lambda_2 = 0$ so $\bigtriangledown f = 0$, a contradiction. Hence, exactly two of a, b, and c must be equal.

Without loss of generality, assume $a \neq b = c$. Then the original constraints become

$$a^2 + 2b^2 = 18 \quad \text{and} \quad ab^2 = 4,$$

which yields

$$\begin{aligned} &a^2 + \frac{8}{a} = 18, \\ &a^3 - 18a + 8 = 0, \\ &(a-4)(a^2 + 4a - 2) = 0. \end{aligned}$$

Hence, $a = 4$ or $a = -2 \pm \sqrt{6}$. Since $a > 0$, we obtain $a = 4$ or $\sqrt{6} - 2$.

If $a = 4$, then $b = c = 1$, and if $a = \sqrt{6} - 2$, then $b = c = \sqrt{2\sqrt{6} + 4}$. These points then yield the extreme values of

$$f(4,1,1) = 6 \quad \text{and} \quad f(\sqrt{6}-2, \sqrt{2\sqrt{6}+4}, \sqrt{2\sqrt{6}+4}) = 2\sqrt{2\sqrt{6}+4} + \sqrt{6} - 2.$$

Since

$$2\sqrt{2\sqrt{6}+4} + \sqrt{6} - 2 \approx 6.4157 > 6,$$

the results follow. Note that the maximum is attained at the 3 points obtained by permuting the coordinates of $(4, 1, 1)$ and the minimum is attained at the 3 points obtained by permuting the coordinates of

$$(\sqrt{6}-2, \sqrt{2\sqrt{6}+4}, \sqrt{2\sqrt{6}+4}).$$

4169. *Proposed by Michel Bataille.*

Let a, b, c be positive real numbers. Prove that

$$\left(a\sqrt{\frac{b}{a+b}} + b\sqrt{\frac{c}{b+c}} + c\sqrt{\frac{a}{c+a}}\right)\left(b\sqrt{\frac{a+b}{b}} + c\sqrt{\frac{b+c}{c}} + a\sqrt{\frac{c+a}{a}}\right) \le (a+b+c)^2.$$

There were 14 correct solutions submitted. Eight of these solvers independently gave the solution presented here.

Using the Cauchy-Schwarz and arithmetic-harmonic means inequalities, we obtain the following two inequalities:

$$\begin{aligned} a\sqrt{\frac{b}{a+b}} + b\sqrt{\frac{c}{b+c}} + c\sqrt{\frac{a}{c+a}} &= \sqrt{a}\sqrt{\frac{ab}{a+b}} + \sqrt{b}\sqrt{\frac{bc}{b+c}} + \sqrt{c}\sqrt{\frac{ca}{c+a}} \\ &= \sqrt{a+b+c}\sqrt{\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}} \\ &\le \sqrt{a+b+c}\sqrt{\frac{a+b}{4} + \frac{b+c}{4} + \frac{c+a}{4}} \\ &= \frac{1}{\sqrt{2}}(a+b+c) \end{aligned}$$

and

$$b\sqrt{\frac{a+b}{b}} + c\sqrt{\frac{b+c}{c}} + a\sqrt{\frac{c+a}{a}} = \sqrt{b}\sqrt{a+b} + \sqrt{c}\sqrt{c+b} + \sqrt{a}\sqrt{c+a}$$
$$\leq \sqrt{a+b+c}\sqrt{(a+b)+(b+c)+(c+a)}$$
$$= \sqrt{2}(a+b+c).$$

Multiplying these inequalities gives the desired result.

Editor's Comment. Nguyen Ngoc Tú used the concavity of the function $\sqrt{x}$ and reduced the problem to establishing that

$$ab(a+b)^{-1} + bc(b+c)^{-1} + ca(c+a)^{-1} \leq (a+b+c)/2.$$

4170. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let $ABCD$ be a circumscribed quadrilateral (that is, a quadrilateral for which an incircle can be constructed) and let P be the intersection point of AC and BD. Let h_a, h_b, h_c and h_d denote the distances from P to AB, BC, CD and DA, respectively. Prove that

$$\frac{AB \cdot CD}{AD \cdot BC} = \frac{h_b + h_d}{h_a + h_c}.$$

We received nine submissions, all correct, and will feature two of them. The first is typical of those that reduced the problem to a fairly recent result; the second is provided for those readers who prefer to see the details.

Solution 1, by Adnan Ali.

Let θ be the angle between the diagonals AC and BD. Then observe that we have the following relations:

$$AB \cdot h_a = PA \cdot PB \sin\theta,$$
$$BC \cdot h_b = PB \cdot PC \sin\theta,$$
$$CD \cdot h_c = PC \cdot PD \sin\theta,$$
$$DA \cdot h_d = PD \cdot PA \sin\theta.$$

It follows that

$$\frac{AB \cdot CD}{AD \cdot BC} = \frac{h_b \cdot h_d}{h_a \cdot h_c}.$$

The desired result is thereby reduced to proving

$$\frac{h_b \cdot h_d}{h_a \cdot h_c} = \frac{h_b + h_d}{h_a + h_c}.$$

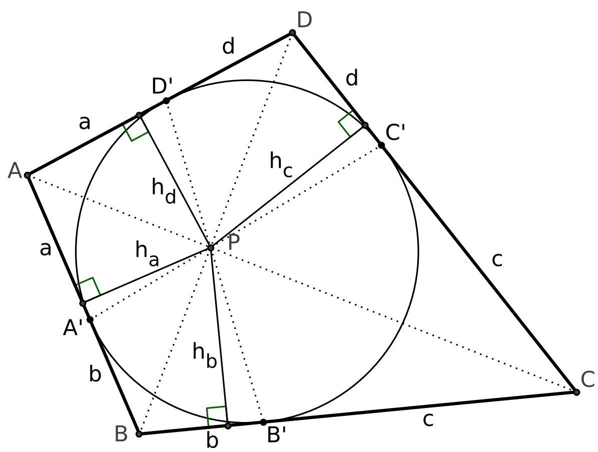
But this is an immediate consequence of a known theorem, namely, a convex quadrilateral $ABCD$ has an incircle if and only if the altitudes h_a, h_b, h_c, h_d from the intersection point of the diagonals to each of the four sides satisfy

$$\frac{1}{h_a} + \frac{1}{h_c} = \frac{1}{h_b} + \frac{1}{h_d}.$$

Proofs of the theorem can be found in [2], for example. Moreover, this result was Problem 5 on the 2015 Indian National Mathematics Olympiad.

Solution 2 is a composite of similar solutions from Oliver Geupel and John G. Heuver.

Let A', B', C', and D' be the points where the lines AB, BC, CD, and DA, respectively, touch the incircle Γ, and denote the tangent lengths by $a = AA' = AD'$, $b = BB' = BA'$, $c = CC' = CB'$, and $d = DD' = DA'$. A degenerate version of Brianchon's theorem tells us that the chords $A'C'$ and $B'D'$ also pass through P. Here is an alternative proof of this claim: Let Q be the intersection point of the lines AC and $A'C'$.



The tangents AB and CD to Γ are symmetric with respect to the perpendicular bisector of the chord $A'C'$. Hence $\angle QA'A = \angle DC'Q = 180^\circ - \angle QC'C$. By the Sine Law applied to triangles AQA' and CQC' we obtain

$$\frac{AQ}{AA'} = \frac{\sin\angle QA'A}{\sin\angle AQA'} = \frac{\sin\angle QC'C}{\sin\angle CQC'} = \frac{CQ}{CC'};$$

that is, $AQ : CQ = a : c$. Similarly, if R is the intersection point of AC and $B'D'$, we have $AR : CR = a : c$. It follows that $Q = R$ so that the lines AC, $A'C'$, and $B'D'$ are concurrent at point $Q = R$. Similarly, the lines BD, $A'C'$ and $B'D'$ are concurrent at $Q = R$. Thus, the lines AC, BD, $A'C'$, and $B'D'$ are concurrent at $P = Q = R$. We deduce, therefore, that

$$\frac{AP}{PC} = \frac{a}{c} \quad \text{and} \quad \frac{BP}{PD} = \frac{b}{d}.$$

It follows that $\frac{[ABP]}{[BCP]} = \frac{AP}{PC} = \frac{a}{c}$, where we have used square brackets for the area of a triangle. Analogous results involving the other relevant triangles give us

$$\frac{[ABP]}{ab} = \frac{[BCP]}{bc} = \frac{[CDP]}{cd} = \frac{[DAP]}{da}. \tag{1}$$

Note that $2[ABP] = (a+b)h_a$, so that

$$h_a = \frac{2[ABP]}{a+b} = \frac{2kab}{a+b}, \tag{2}$$

where we have set k equal to the common ratio in (1). Similar formulas hold for the remaining triangles. We finally obtain

$$\begin{aligned} \frac{AB \cdot CD}{AD \cdot BC} &= \frac{(a+b)(c+d)}{(d+a)(b+c)} \\ &= \frac{\frac{1}{(b+c)} \cdot \frac{1}{(d+a)}}{\frac{1}{(a+b)} \cdot \frac{1}{(c+d)}} \\ &= \frac{\frac{bc}{b+c} + \frac{da}{d+a}}{\frac{ab}{a+b} + \frac{cd}{c+d}} = \frac{h_b + h_d}{h_a + h_c}, \end{aligned}$$

which completes the proof.

Editor's comments. Note that equation (2) (together with the corresponding equations for h_b, h_c, h_d) lead immediately to a proof of the theorem mentioned in the first solution. Other proofs of the theorem can be found in [1], where Josefsson traces the result back to a 1995 problem in the Russian Journal *Kvant* [3]. He reports that it also appeared in [4], and as a problem in the 4th stage of the 48th German Mathematical Olympiad (2009).

References

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[2] N. Minculete, Characterizations of a Tangential Quadrilateral, *Forum Geometricorum*, **9** (2009) 113–118.

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[4] A. Zaslavsky, Problem M1887, *Kvant* (in Russian) no. 3 (2004) p. 19.