THE OLYMPIAD CORNER

No. 355

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The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **March 1, 2018***.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC341. There are 30 teams in the NBA and every team plays 82 games in the year. Owners of the NBA teams want to divide all teams into Western and Eastern Conferences (not necessarily equally), such that the number of games between teams from different conferences is half of number of all games. Can they do it?

OC342. Consider the second-degree polynomial $P(x) = 4x^2 + 12x - 3015$. Define the sequence of polynomials $P_1(x) = \frac{P(x)}{2016}$ and $P_{n+1}(x) = \frac{P(P_n(x))}{2016}$ for every integer $n \geq 1$.

a) Show that there exists a real number r such that $P_n(r) < 0$ for every positive integer n.

b) For how many integers m does $P_n(m) < 0$ hold for infinitely many positive integers n?

OC343. Determine all pairs of positive integers (a, n) with $a \geq n \geq 2$ for which $(a+1)^n + a - 1$ is a power of 2.

OC344. Find all $a \in \mathbb{R}$ such that there exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying

- $f(1) = 2016$;
- $f(x + y + f(y)) = f(x) + ay \quad \forall x, y \in \mathbb{R}$.

OC345. Let $\triangle ABC$ be an acute triangle, and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$. Lines $\overleftrightarrow{I_B F}$ and $\overleftrightarrow{I_C E}$ meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

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OC341. La NBA compte 30 équipes et chaque équipe joue 82 matchs au cours de la saison régulière. Les propriétaires des équipes de la NBA aimeraient placer les équipes dans la conférence de l'est ou dans la conférence de l'ouest (sans que les conférences aient nécessairement un nombre égal d'équipes), de manière que le nombre de matchs entre des équipes de chaque conférence soit égal à la moitié du nombre total de matchs. Peuvent-ils réussir à le faire?

OC342. On considère le polynôme du second degré $P(x) = 4x^2 + 12x - 3015$. On définit une suite de polynômes comme suit:

$$P_1(x) = \frac{P(x)}{2016} \text{ et } P_{n+1}(x) = \frac{P(P_n(x))}{2016} \text{ pour tout entier } n \ (n \geq 1).$$

a) Démontrer qu'il existe un réel r tel que $P_n(r) < 0$ pour tous entiers n strictement positifs.

b) Combien y a-t-il d'entiers m tels que $P_n(m) < 0$ pour un nombre infini d'entiers n strictement positifs?

OC343. Déterminer tous les couples d'entiers (a, n) $(a \geq n \geq 2)$ pour lesquels la valeur de $(a + 1)^n + a - 1$ est une puissance de 2.

OC344. Déterminer tous les réels a pour lesquels il existe une fonction f $(f : \mathbb{R} \to \mathbb{R})$ qui satisfait à

- $f(1) = 2016$,
- $f(x + y + f(y)) = f(x) + ay \quad \forall x, y \in \mathbb{R}$.

OC345. Soit ABC un triangle acutangle et O le centre du cercle circonscrit au triangle. Soit I_B et I_C les centres respectifs des cercles exinscrits dans les angles B et C. Des points E et Y sont choisis sur le segment AC tels que $\angle ABY = \angle CBY$ et $BE \perp AC$. De même, des points F et Z sont choisis sur le segment AB tels que $\angle ACZ = \angle BCZ$ et $CF \perp AB$. Les droites I_BF et I_CE se coupent en P. Démontrer que les segments PO et YZ sont perpendiculaires.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(5), p. 202–204.

OC281. Find all polynomials $P(x)$ with real coefficients such that

$$P(P(x)) = (x^2 + x + 1) \cdot P(x)$$

where $x \in \mathbb{R}$.

Originally problem 3 from the Junior Level of the 2015 Azerbaijan National Olympiad.

We received 8 correct submissions. We present the solution by Pedro Acosta.

First, observe that the zero polynomial works so throughout assume that P is not the zero polynomial.

Let $d = \deg P$. Then $\deg(P \circ P) = d^2$ and $\deg(P(x) \cdot (x^2 + x + 1)) = d+2$. Thus $d^2 = d + 2 \Rightarrow d = 2$.

Now let α be a root of P. Then

$$P(0) = P(P(\alpha)) = P(\alpha) \cdot (\alpha^2 + \alpha + 1) = 0,$$

so 0 is a root of P. Furthermore, from comparing leading coefficients we see that P must be monic. Thus $P(x) = x^2 - ax$ for some real number a; it suffices to compute all possible values of a.

To do this, note that if ω is a root of $x^2 + x + 1$, then $P(P(\omega)) = 0$. Thus $P(\omega) = 0$ or $P(\omega) = a$. In the former case, we must have $P(x) = x^2 - \omega x$, but this contradicts the fact that the coefficients of the polynomial are real. In the latter case, we then have

$$\omega^2 - a\omega = a \quad \Longrightarrow \quad \omega^2 = a(\omega + 1) = -a\omega^2,$$

i.e. $a = -1$. As a result, $P(x) = x^2 + x$ is the only possible nonzero polynomial P, and one can check that it works.

Editor's Comment: just a gentle reminder to solvers to remember to check the zero polynomial when using polynomial degree arguments.

OC282. Let x, y, z be three nonzero real numbers satisfying $x + y + z = xyz$. Prove that

$$\sum \left(\frac{x^2 - 1}{x}\right)^2 \geq 4.$$

Originally problem 1 of the 2015 Iranian Mathematical Olympiad.

We received 5 correct submissions. We present the solution by Daniel Sitaru.

Let A, B, C be the angles of ΔABC and let $x = \tan A, y = \tan B, z = \tan C$ with sides a, b, c and semi-perimeter S, valid since:

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C \Leftrightarrow x + y + z = xyz.$$

Then,

$$\begin{aligned}\sum\Big(\frac{x^2-1}{x}\Big)^2 \geq 4 &\Leftrightarrow \sum\Big(\frac{\tan^2 A - 1}{\tan A}\Big)^2 \geq 4\\ &\Leftrightarrow \sum(2\cot(2A))^2 \geq 1\\ &\Leftrightarrow \sum \cot^2(2A) \geq 1.\end{aligned}$$

Now, notice that

$$\sqrt{\frac{\sum \cot^2(2A)}{3}} \geq \frac{\sum \cot(2A)}{3}$$

and so

$$\begin{aligned}\sum \cot^2(2A) &\geq \frac{(\sum \cot(2A))^2}{3}\\ &= \frac{1}{3}\Big(\frac{b^2+c^2-a^2}{4S} + \frac{a^2+b^2-c^2}{4S} + \frac{a^2+c^2-b^2}{4S}\Big)^2\\ &= \frac{1}{3}\Big(\frac{a^2+b^2+c^2}{4S}\Big)^2.\end{aligned}$$

It suffices to show this last term above is at least one. However, this follows from the Ionescu-Weitzenbock's inequality which states that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S$$

and hence

$$(a^2+b^2+c^2) \geq 48S^2 \Leftrightarrow .\frac{1}{3}\Big(\frac{a^2+b^2+c^2}{4S}\Big)^2 \geq 1$$

OC283. In isosceles $\triangle ABC$, $AB = AC$, I is its incenter, D is a point inside $\triangle ABC$ such that I, B, C, D are concyclic. The line through C parallel to BD meets AD at E. Prove that $CD^2 = BD \cdot CE$.

Originally problem 2 from Part B of the 2015 China Second Round Olympiad.

We received 2 correct submissions. We present the solution by Andrea Fanchini.

We use barycentric coordinates and the usual Conway's notations with reference to triangle ABC. First of all we have

$$b = c, \qquad S_B = S_C = \frac{a^2}{2}, \qquad I(a:b:b)$$

Equation of a generic circle is

$$a^2yz + b^2zx + c^2xy - (x+y+z)(px+qy+rz) = 0.$$

If this circle passes through B, C, I, we obtain the three conditions

$$q = 0, \qquad r = 0, \qquad p = b^2,$$

so this circle has equation

$$a^2yz + b^2zx + c^2xy - (x + y + z)b^2x = 0 \quad \Rightarrow \quad a^2yz - b^2x^2 = 0.$$

Then a generic point D on this circle and inside $\triangle ABC$ has coordinates $D(a\sqrt{t} : b : bt)$ where t is a parameter.

Lines AD and BD have equations

$$AD : ty - z = 0, \qquad BD : btx - a\sqrt{t}z = 0.$$

The line through C parallel to BD is

$$CBD_\infty : (a\sqrt{t} + bt)x + a\sqrt{t}y = 0,$$

therefore point E is the intersection between AD and CBD_∞

$$E = AD \cap CBD_\infty = (a\sqrt{t} : -a\sqrt{t} - bt : -at\sqrt{t} - bt^2).$$

Now we can calculate the distances CD, BD, CE that are

$$CD^2 = \frac{a^2b}{a\sqrt{t} + bt + b}, \qquad BD = \sqrt{\frac{a^2bt}{a\sqrt{t} + bt + b}}, \qquad CE = \sqrt{\frac{a^2b}{t(a\sqrt{t} + bt + b)}}$$

and we are done.

OC284. A positive integer n is given. If there exists sets $F_1, F_2, \ldots, F_m$ satisfying the following conditions, prove that $m \leq n$.

1. For all $1 \leq i \leq m$, $F_i \subseteq \{1, 2, \cdots, n\}$
2. For all $1 \leq i < j \leq m$, $\min(|F_i - F_j|, |F_j - F_i|) = 1$

Originally problem 3 from day 2 of the 2015 Korea National Olympiad.

We present the solution by Steven Chow. There were no other submissions.

It shall be proved using mathematical induction on n.

If $n = 1$, then $m \leq 1 \leq n$. (From condition 2, for all $1 \leq j \leq m$, $F_j \neq \emptyset$.)

If $n = 2$, then $m \leq 2 = n$: $\min\left(|\{1\}\backslash\{1, 2\}|, |\{1, 2\}\backslash\{1\}|\right) = \min\,(0, 1) = 0 \neq 1$.

Assume for the sake of contradiction that $m \geq k + 1$. For all $1 \leq j \leq m$, $1 \leq |F_j| \leq k$, so there exists $1 \leq a < b \leq m$ such that $|F_a| = |F_b|$. Since we have $\min\left(|F_a\backslash F_b|, |F_b\backslash F_a|\right) = 1$, therefore $|F_a\backslash(F_a \cap F_b)| = |F_b\backslash(F_a \cap F_b)| = 1$, so there exists $1 \leq r, s \leq k$ such that $r \in F_a\backslash(F_a \cap F_b)$ and $s \in F_b\backslash(F_a \cap F_b)$.

If there exists $1 \leq c \leq m$ such that $c \neq a, b$ and $r \in F_c$, $s \notin F_c$, then since $\min\left(|F_b \backslash F_c|, |F_c \backslash F_b|\right) = 1$, it follows that $F_b \backslash F_c = \{s\}$ and $F_c \backslash F_b = \{r\}$, so $F_b \cap F_c = F_a \cap F_b$ and $F_c = F_a$ which is a contradiction. Therefore there does not exist $1 \leq c \leq m$ such that $c \neq a, b$ and $r \in F_c$, $s \notin F_c$, and similarly there does not exist $1 \leq c \leq m$ such that $c \neq a, b$ and $r \notin F_c$, $s \in F_c$.

Therefore for all integers $1 \leq j \leq m$ such that $j \neq a, b$, either $r, s \notin F_j$ or $r, s \in F_j$, so the sets F_j not containing r or s satisfy conditions 1 and 2.

From the induction hypothesis, we have $m \leq 2 + (k-2) = k$. A contradiction.

Therefore $m \leq k$, and if $n = k$, then the statement is true. Therefore $m \leq n$.

OC285. Show that from a set of 11 square integers one can select six numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that $a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}$.

Originally problem 6 of the 2015 India National Olympiad.

We received 3 correct submissions. We present the solution by Somasundaram Muralidharan.

When the square of an integer is divided by 12, the possible remainders are $0, 1, 4, 9$. Out of the 11 square numbers, if there are 6 or more numbers that leave the same remainder r when divided by 12, then clearly we can select 2 sets of three numbers from them such that their sums leave the same remainder $3r$ when divided by 12. Suppose that for any remainder r there are no more than 5 numbers n such that n^2 leaves the remainder r when divided by 12. Firstly, suppose that there is a remainder r_1 for which there are 4 numbers n such that n^2 leaves a remainder r_1 when divided by 12. Out of the remaining 7 numbers, there is at least one remainder $r_2 \neq r_1$ such that there are at least two n' whose squares leave remainder r_2. Now we can choose a, b, d, e whose squares leave remainder r_1 and c, f whose squares leave remainder r_2. Then $a^2+b^2+c^2 = (2r_1+r_2) \mod 12 = d^2+e^2+f^2$. We are now left with the case in which the split of the numbers is $3, 3, 3, 2$ for the remainders. In this case we can take numbers such that

$$a^2 = 0 \mod 12, \quad b^2 = 9 \mod 12, \quad c^2 = 9 \mod 12,$$
$$d^2 = 1 \mod 12, \quad e^2 = 1 \mod 12, \quad f^2 = 4 \mod 12.$$

Then we have

$$a^2 + b^2 + c^2 = 6 \mod 12,$$
$$d^2 + e^2 + f^2 = 6 \mod 12.$$

This completes the proof.

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Editor's Comments. Congratulations to Steven Chow who solved all Olympiad Corner problems correctly.