OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(5), p. 202–204.

OC281. Find all polynomials $P(x)$ with real coefficients such that

$$P(P(x)) = (x^2 + x + 1) \cdot P(x)$$

where $x \in \mathbb{R}$.

Originally problem 3 from the Junior Level of the 2015 Azerbaijan National Olympiad.

We received 8 correct submissions. We present the solution by Pedro Acosta.

First, observe that the zero polynomial works so throughout assume that P is not the zero polynomial.

Let $d = \deg P$. Then $\deg(P \circ P) = d^2$ and $\deg(P(x) \cdot (x^2 + x + 1)) = d+2$. Thus $d^2 = d + 2 \Rightarrow d = 2$.

Now let α be a root of P. Then

$$P(0) = P(P(\alpha)) = P(\alpha) \cdot (\alpha^2 + \alpha + 1) = 0,$$

so 0 is a root of P. Furthermore, from comparing leading coefficients we see that P must be monic. Thus $P(x) = x^2 - ax$ for some real number a; it suffices to compute all possible values of a.

To do this, note that if ω is a root of $x^2 + x + 1$, then $P(P(\omega)) = 0$. Thus $P(\omega) = 0$ or $P(\omega) = a$. In the former case, we must have $P(x) = x^2 - \omega x$, but this contradicts the fact that the coefficients of the polynomial are real. In the latter case, we then have

$$\omega^2 - a\omega = a \quad \Longrightarrow \quad \omega^2 = a(\omega + 1) = -a\omega^2,$$

i.e. $a = -1$. As a result, $P(x) = x^2 + x$ is the only possible nonzero polynomial P, and one can check that it works.

Editor's Comment: just a gentle reminder to solvers to remember to check the zero polynomial when using polynomial degree arguments.

OC282. Let x, y, z be three nonzero real numbers satisfying $x + y + z = xyz$. Prove that

$$\sum \left(\frac{x^2 - 1}{x} \right)^2 \geq 4.$$

Originally problem 1 of the 2015 Iranian Mathematical Olympiad.

We received 5 correct submissions. We present the solution by Daniel Sitaru.

Let A, B, C be the angles of ΔABC and let $x = \tan A, y = \tan B, z = \tan C$ with sides a, b, c and semi-perimeter S, valid since:

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C \Leftrightarrow x + y + z = xyz.$$

Then,

$$\begin{aligned}\sum\Big(\frac{x^2-1}{x}\Big)^2 \geq 4 &\Leftrightarrow \sum\Big(\frac{\tan^2 A - 1}{\tan A}\Big)^2 \geq 4 \\ &\Leftrightarrow \sum(2\cot(2A))^2 \geq 1 \\ &\Leftrightarrow \sum \cot^2(2A) \geq 1.\end{aligned}$$

Now, notice that

$$\sqrt{\frac{\sum \cot^2(2A)}{3}} \geq \frac{\sum \cot(2A)}{3}$$

and so

$$\begin{aligned}\sum \cot^2(2A) &\geq \frac{(\sum \cot(2A))^2}{3} \\ &= \frac{1}{3}\Big(\frac{b^2+c^2-a^2}{4S} + \frac{a^2+b^2-c^2}{4S} + \frac{a^2+c^2-b^2}{4S}\Big)^2 \\ &= \frac{1}{3}\Big(\frac{a^2+b^2+c^2}{4S}\Big)^2.\end{aligned}$$

It suffices to show this last term above is at least one. However, this follows from the Ionescu-Weitzenbock's inequality which states that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S$$

and hence

$$(a^2+b^2+c^2) \geq 48S^2 \Leftrightarrow .\frac{1}{3}\Big(\frac{a^2+b^2+c^2}{4S}\Big)^2 \geq 1$$

OC283. In isosceles $\triangle ABC$, $AB = AC$, I is its incenter, D is a point inside $\triangle ABC$ such that I, B, C, D are concyclic. The line through C parallel to BD meets AD at E. Prove that $CD^2 = BD \cdot CE$.

Originally problem 2 from Part B of the 2015 China Second Round Olympiad.

We received 2 correct submissions. We present the solution by Andrea Fanchini.

We use barycentric coordinates and the usual Conway's notations with reference to triangle ABC. First of all we have

$$b = c, \qquad S_B = S_C = \frac{a^2}{2}, \qquad I(a : b : b)$$

Equation of a generic circle is

$$a^2yz + b^2zx + c^2xy - (x+y+z)(px+qy+rz) = 0.$$

If this circle passes through B, C, I, we obtain the three conditions

$$q = 0, \qquad r = 0, \qquad p = b^2,$$

so this circle has equation

$$a^2yz + b^2zx + c^2xy - (x+y+z)b^2x = 0 \quad \Rightarrow \quad a^2yz - b^2x^2 = 0.$$

Then a generic point D on this circle and inside $\triangle ABC$ has coordinates $D(a\sqrt{t} : b : bt)$ where t is a parameter.

Lines AD and BD have equations

$$AD : ty - z = 0, \qquad BD : btx - a\sqrt{t}z = 0.$$

The line through C parallel to BD is

$$CBD_\infty : (a\sqrt{t} + bt)x + a\sqrt{t}y = 0,$$

therefore point E is the intersection between AD and CBD_∞

$$E = AD \cap CBD_\infty = (a\sqrt{t} : -a\sqrt{t} - bt : -at\sqrt{t} - bt^2).$$

Now we can calculate the distances CD, BD, CE that are

$$CD^2 = \frac{a^2b}{a\sqrt{t} + bt + b}, \qquad BD = \sqrt{\frac{a^2bt}{a\sqrt{t} + bt + b}}, \qquad CE = \sqrt{\frac{a^2b}{t(a\sqrt{t} + bt + b)}}$$

and we are done.

OC284. A positive integer n is given. If there exists sets $F_1, F_2, \ldots, F_m$ satisfying the following conditions, prove that $m \leq n$.

1. For all $1 \leq i \leq m$, $F_i \subseteq \{1, 2, \cdots, n\}$
2. For all $1 \leq i < j \leq m$, $\min(|F_i - F_j|, |F_j - F_i|) = 1$

Originally problem 3 from day 2 of the 2015 Korea National Olympiad.

We present the solution by Steven Chow. There were no other submissions.

It shall be proved using mathematical induction on n.

If $n = 1$, then $m \leq 1 \leq n$. (From condition 2, for all $1 \leq j \leq m$, $F_j \neq \emptyset$.)

If $n = 2$, then $m \leq 2 = n$: $\min\,(|\{1\}\backslash\{1,2\}|, |\{1,2\}\backslash\{1\}|) = \min\,(0,1) = 0 \neq 1$.

Assume for the sake of contradiction that $m \geq k+1$. For all $1 \leq j \leq m$, $1 \leq |F_j| \leq k$, so there exists $1 \leq a < b \leq m$ such that $|F_a| = |F_b|$. Since we have $\min\,(|F_a \backslash F_b|, |F_b \backslash F_a|) = 1$, therefore $|F_a \backslash (F_a \cap F_b)| = |F_b \backslash (F_a \cap F_b)| = 1$, so there exists $1 \leq r, s \leq k$ such that $r \in F_a \backslash (F_a \cap F_b)$ and $s \in F_b \backslash (F_a \cap F_b)$.

If there exists $1 \le c \le m$ such that $c \ne a, b$ and $r \in F_c$, $s \notin F_c$, then since $\min\left(|F_b \backslash F_c|, |F_c \backslash F_b|\right) = 1$, it follows that $F_b \backslash F_c = \{s\}$ and $F_c \backslash F_b = \{r\}$, so $F_b \cap F_c = F_a \cap F_b$ and $F_c = F_a$ which is a contradiction. Therefore there does not exist $1 \le c \le m$ such that $c \ne a, b$ and $r \in F_c$, $s \notin F_c$, and similarly there does not exist $1 \le c \le m$ such that $c \ne a, b$ and $r \notin F_c$, $s \in F_c$.

Therefore for all integers $1 \le j \le m$ such that $j \ne a, b$, either $r, s \notin F_j$ or $r, s \in F_j$, so the sets F_j not containing r or s satisfy conditions 1 and 2.

From the induction hypothesis, we have $m \le 2 + (k-2) = k$. A contradiction.

Therefore $m \le k$, and if $n = k$, then the statement is true. Therefore $m \le n$.

OC285. Show that from a set of 11 square integers one can select six numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that $a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}$.

Originally problem 6 of the 2015 India National Olympiad.

We received 3 correct submissions. We present the solution by Somasundaram Muralidharan.

When the square of an integer is divided by 12, the possible remainders are $0, 1, 4, 9$. Out of the 11 square numbers, if there are 6 or more numbers that leave the same remainder r when divided by 12, then clearly we can select 2 sets of three numbers from them such that their sums leave the same remainder $3r$ when divided by 12. Suppose that for any remainder r there are no more than 5 numbers n such that n^2 leaves the remainder r when divided by 12. Firstly, suppose that there is a remainder r_1 for which there are 4 numbers n such that n^2 leaves a remainder r_1 when divided by 12. Out of the remaining 7 numbers, there is at least one remainder $r_2 \ne r_1$ such that there are at least two n' whose squares leave remainder r_2. Now we can choose a, b, d, e whose squares leave remainder r_1 and c, f whose squares leave remainder r_2. Then $a^2 + b^2 + c^2 = (2r_1 + r_2) \mod 12 = d^2 + e^2 + f^2$. We are now left with the case in which the split of the numbers is $3, 3, 3, 2$ for the remainders. In this case we can take numbers such that

$$\begin{aligned} a^2 &= 0 \mod 12, \quad b^2 = 9 \mod 12, \quad c^2 = 9 \mod 12, \\ d^2 &= 1 \mod 12, \quad e^2 = 1 \mod 12, \quad f^2 = 4 \mod 12. \end{aligned}$$

Then we have

$$\begin{aligned} a^2 + b^2 + c^2 &= 6 \mod 12, \\ d^2 + e^2 + f^2 &= 6 \mod 12. \end{aligned}$$

This completes the proof.

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Editor's Comments. Congratulations to Steven Chow who solved all Olympiad Corner problems correctly.