FOCUS ON...

No. 27
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Some relations in the triangle (I)

Introduction

Relations between the elements of a triangle intervene in ***Crux*** problems – and solutions – quite often, and that's an understatement! Every solver wanting to establish an identity or an inequality involving those elements should have a number of these relations in her/his toolbox: the Laws of Sines and Cosines of course, but also classical relations such as $a = b\cos C + c\cos B$ or $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

However, even the most common of these relations form a vast subject; for instance, the compendium of them in **[1]** spreads over some twenty pages! The modest goal of this number and the next one is to present a selection of less familiar relations chosen because of their aesthetic qualities and/or their applications to problems.

Here and in what follows, we use the standard notations as they can be found in **[2]** (where α, β, γ are preferred to A, B, C for the angles of the triangle, though).

About the differences of angles $B - C, C - A$ and $A - B$

We begin with the surprising relation

$$a\cos(B-C) + b\cos(C-A) + c\cos(A-B) = 2(a\cos A + b\cos B + c\cos C). \quad (1)$$

The proof is easy: we have that $a\sin C = c\sin A$ (from the Law of Sines) and $b = a\cos C + c\cos A$, therefore

$$\begin{aligned} a\cos(B-C) - b\cos B &= a\cos B\cos C + a\sin B\sin C - b\cos B \\ &= \cos B(a\cos C - b) + c\sin A\sin B \\ &= -c\cos A\cos B + c\sin A\sin B \\ &= -c\cos(A+B) \\ &= c\cos C. \end{aligned}$$

Hence

$$a\cos(B-C) = b\cos B + c\cos C.$$

With similar results for $b\cos(C-A)$ and $c\cos(A-B)$, the desired equality is deduced at once.

With the help of the Law of Cosines, (1) leads to

$$\begin{aligned} &a\cos(B-C) + b\cos(C-A) + c\cos(A-B) \\ &\qquad = \frac{2(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4}{abc} \end{aligned} \quad (2)$$

so that

$$a\cos(B-C)+b\cos(C-A)+c\cos(A-B)=\frac{16F^2}{4RF}=\frac{4F}{R}=\frac{abc}{R^2}=\frac{4rs}{R}.$$

Here are two applications. First, the latter shows that $\frac{4rs}{R}\leq a+b+c=2s$ and we obtain Euler's inequality $R\geq 2r$ in a rather oblique way!

Second, (1) yields a neat solution to problem **2546** [2000 : 237 ; 2001 : 343]

> Prove that triangle ABC is equilateral if and only if
>
> $$a\cos(B-C)+b\cos(C-A)+c\cos(A-B)=\frac{a^4+b^4+c^4}{abc},$$

for (2) shows that the given relation is equivalent to

$$2(a^4+b^4+c^4)=2(a^2b^2+b^2c^2+c^2a^2),$$

that is, to

$$(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2=0,$$

which clearly holds if and only if $a=b=c$.

In the same vein, we will prove the following beautiful formula

$$a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)=3abc. \qquad (3)$$

To this aim, we start with expressions of the area F that deserve to be better known:

$$4F=a^2\sin 2B+b^2\sin 2A=b^2\sin 2C+c^2\sin 2B=c^2\sin 2A+a^2\sin 2C. \qquad (4)$$

This follows, for example, from

$$a^2\sin 2B+b^2\sin 2A=2a^2\sin B\cos B+2ab\sin B\cos A=2ac\sin B=4F.$$

Formula (3) is deduced smoothly once we have noticed that

$$\begin{aligned}a^3\cos(B-C)&=a^2\cdot 2R\sin A\cos(B-C)\\&=2Ra^2\sin(B+C)\cos(B-C)\\&=Ra^2(\sin 2B+\sin 2C).\end{aligned}$$

Then, using (4),

$$\begin{aligned}&\sum_{\text{cyclic}}a^3\cos(B-C)\\&=R(a^2\sin 2B+a^2\sin 2C+b^2\sin 2C+b^2\sin 2A+c^2\sin 2A+c^2\sin 2B)\\&=R(4F+4F+4F)\\&=12RF\\&=3abc.\end{aligned}$$

About the cosines of $\dfrac{B-C}{2}, \dfrac{C-A}{2}$ and $\dfrac{A-B}{2}$

We will consider two expressions of $\cos\left(\frac{B-C}{2}\right)$ and apply them to past problems.

The first one is readily obtained:

$$\cos\left(\frac{B-C}{2}\right) = \frac{b+c}{a}\cdot \sin\frac{A}{2}. \tag{5}$$

Indeed, because $\cos\frac{A}{2} = \sin(\frac{B+C}{2})$, we have

$$\frac{\cos(\frac{B-C}{2})}{\sin\frac{A}{2}} = \frac{2\sin(\frac{B+C}{2})\cos(\frac{B-C}{2})}{2\cos\frac{A}{2}\sin\frac{A}{2}} = \frac{\sin B+\sin C}{\sin A} = \frac{b+c}{a}.$$

Relation (5) and similar relations provide a quick and easy solution to problem **2717** [2002 : 112; 2003 : 119], which states

For any triangle ABC, prove that

$$8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right).$$

This immediately follows from (5):

$$\begin{aligned}\frac{\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} &= \left(\frac{b}{a}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{a}{b}\right)\left(\frac{a}{c}+\frac{b}{c}\right)\\ &= 2+\left(\frac{b}{a}+\frac{a}{b}\right)+\left(\frac{c}{b}+\frac{b}{c}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)\\ &\ge 8,\end{aligned}$$

since $x+\frac{1}{x}\ge 2$ for all positive real numbers x.

Note that the inequality rewrites as

$$\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right) \ge \frac{2r}{R}. \tag{6}$$

From (6) we easily derive the related inequality

$$\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{C-A}{2}\right) \ge 1+\frac{4r}{R}. \tag{7}$$

Proof. Since

$$\cos\left(\frac{A-B}{2}\right) \ge \cos^2\left(\frac{A-B}{2}\right) = \frac{1}{2}+\frac{1}{2}\cos(A-B)$$

it is sufficient to show that

$$\cos(A-B)+\cos(B-C)+\cos(C-A) \ge \frac{8r}{R}-1.$$

But, when $x+y+z=0$, we have

$$\cos x+\cos y+\cos z=4\cos\frac{x}{2}\cos\frac{y}{2}\cos\frac{z}{2}-1,$$

hence the latter inequality readily transforms into (6) and therefore is true.

It is interesting to notice that this is a variant of proof of an old ***Crux*** inequality, namely

$$\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{C-A}{2}\right)\geq 4(\cos A+\cos B+\cos C)-3,$$

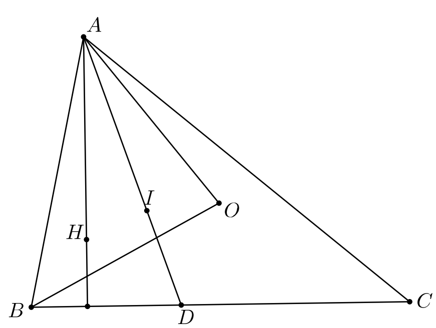
proposed in problem **696** [1981 : 302 ; 1982 : 316] (taking into account the known formula $\cos A+\cos B+\cos C=1+\frac{r}{R}$).

The following additional formula involves the distance IA from the incenter I to the vertex A:

$$\cos\left(\frac{B-C}{2}\right)=\frac{IA}{2R}+\frac{r}{IA}.$$

To prove this relation, we introduce the circumcentre O of ABC and observe that in the case when $B\geq C$, we have $C\leq 90^\circ$ and

$$\angle BAO=\angle OBA=\frac{180^\circ-\angle AOB}{2}=90^\circ-C.$$



In consequence,

$$\angle IAO=\angle BAO-\angle BAI=90^\circ-C-\frac{A}{2}=\frac{B-C}{2}.$$

If $C>B$, we obtain $\angle IAO=\frac{C-B}{2}$ and in either case

$$\cos\left(\frac{B-C}{2}\right)=\frac{IA^2+R^2-IO^2}{2IA\cdot R}=\frac{IA}{2R}+\frac{r}{IA}$$

since $IO^2=R^2-2rR$ (Euler's formula).

Note that an application of the arithmetic-geometric mean inequality yields

$$\cos\left(\frac{B-C}{2}\right)\geq 2\sqrt{\frac{IA}{2R}\cdot\frac{r}{IA}}=2\sqrt{\frac{r}{2R}}$$

or

$$\cos^2\left(\frac{B-C}{2}\right) \ge \frac{2r}{R},$$

the inequality to be proved in problem **2382** [1998 : 425 ; 1999 : 440].

Our second part will offer relations involving the altitudes, exradii, *etc*. By way of transition, let us remark that if H is the orthocentre of ΔABC, then $\angle IAO = \angle IAH$ (recall that the line AO and the altitude from vertex A are symmetric in the angle bisector of $\angle BAC$). We deduce that $\cos\left(\frac{B-C}{2}\right) = \frac{h_a}{w_a}$ where, following the notations of **[2]**, $w_a = AD$ is the length of the angle bisector of $\angle BAC$ (see figure above). Therefore, relations (6) and (7) yield

$$\frac{h_a h_b h_c}{w_a w_b w_c} \ge \frac{2r}{R} \qquad \text{and} \qquad \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \ge 1 + \frac{4r}{R}.$$

Exercises

1.

a) Establish the formula $\sin 2A + \sin 2B + \sin 2C = \dfrac{abc}{2R^3}$ and deduce an expression of $a^2 \sin 2A + b^2 \sin 2B + c^2 \sin 2C$.

b) Prove that

$$a^3 \cos A + b^3 \cos B + c^3 \cos C = \frac{abc}{2R^2} \cdot (a^2 + b^2 + c^2 - 6R^2)$$

and

$$a \cos^3 A + b \cos^3 B + c \cos^3 C = \frac{abc}{8R^4} \cdot (10R^2 - (a^2 + b^2 + c^2)).$$

c) From the latter, deduce that if ΔABC is not obtuse then

$$a \cos^3 A + b \cos^3 B + c \cos^3 C \le \frac{abc}{4R^2}.$$

(the inequality to be proved in problem **3167** [2006 : 395,397 ; 2007 : 374].)

2. Prove the inequality $\sum\limits_{\text{cyclic}} a \cos \frac{B-C}{2} \ge s\left(1 + \frac{2r}{R}\right)$ (use (5) or give a look at problem **696**) and deduce that

$$\frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c} \ge \frac{1}{2r} + \frac{1}{R}.$$

References

[1] T. Lalesco, *La géométrie du triangle*, J. Gabay, 2003, p. 101-120

[2] O. Bottema *et al.*, *Geometric Inequalities*, Wolters-Noordhoff, 1968, p. 9-10