Crux Mathematicorum

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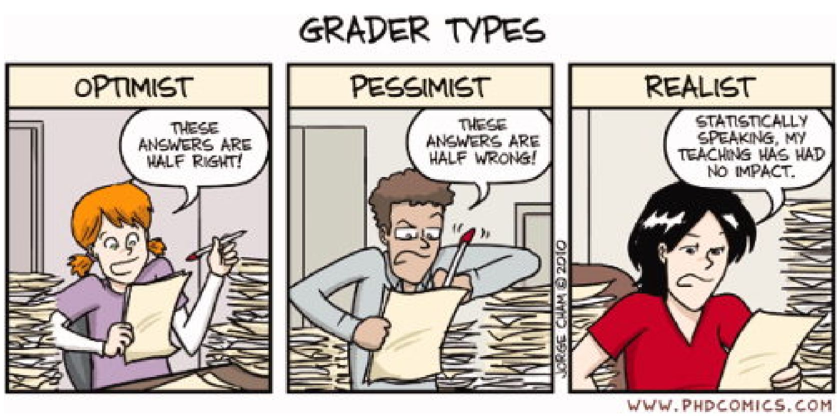
EDITORIAL

I'm currently in full "preparation for teaching" mode. I am preparing lecture materials, making handouts, looking for engaging group activities, finalizing the tests and delaying the inevitable: learning the new online homework system and going through its library of questions. Every technology has its limits and I am particularly mindful of how the students are going to approach the kinds of problems I can assign in the online environment. With limited possibilities for answer types (multiple choice or numerical), the questions tend to be less conceptual and more technical, so it can be very tempting for students to use readily available computer and Internet powers for evil rather than good. They call on the Wolf-man (student slang for Wolfram Alpha) and don't think much about the answer once they obtain it.

In the current problem solving climate, technology also plays a non-trivial role. Nowadays, anyone can get their hands on heavy machinery such as graphing calculators, Wolfram Alpha, Maple, Mathematica, Sage, etc. The first approach to a problem now often consists of investigating it using some suitable software. This has surely enabled us to solve problems that seemed unattainable before: I myself have reaped the benefits of this approach in both my Master and Doctoral work. But while more seasoned problem solvers know the limitations of computer-assisted work and use it appropriately, the new generation is coming with technology that you cannot pry them away from even if it is completely unsuitable for the task at hand.

So I ask them (and you) this: don't forget the beauty of pencil and paper and jotting notes in the margins. It is when the pencil makes marks on the paper than math gets understood. Speaking of pencil and paper: if you enjoy straightedge and compass constructions but would rather not use an eraser, you should visit `http://euclidthegame.com/`. Just don't blame me if you end up wasting a couple of hours.

Kseniya Garaschuk



THE CONTEST CORNER

No. 57

John McLoughlin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **March 1, 2018**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC281. In the Original Six era of the NHL, one particular season, each team played 20 games (each team played the other 5 teams 4 times each). Each game ended as a win, a loss or a tie (there were no 'overtime losses'). At the end of this certain season, the standings were as below. What were all the possible outcomes for Montreal's number of wins X, losses Y and ties Z?

Team	Wins	Losses	Ties
Toronto	2	12	6
Boston	6	10	4
Detroit	7	12	1
New York	7	9	4
Chicago	11	7	2
Montreal	X	Y	Z

CC282. Calculate the value of

$$\left(3^{4/3}-3^{1/3}\right)^3+\left(3^{5/3}-3^{2/3}\right)^3+\left(3^{6/3}-3^{3/3}\right)^3+\cdots+\left(3^{2006/3}-3^{2003/3}\right)^3.$$

CC283. Two bags, Bag A and Bag B, each contain 9 balls. The 9 balls in each bag are numbered from 1 to 9. Suppose one ball is removed randomly from Bag A and another ball from Bag B. If X is the sum of the numbers on the balls left in Bag A and Y is the sum of the numbers of the balls remaining in Bag B, what is the probability that X and Y differ by a multiple of 4?

CC284. Define the function $f(x)$ to be the largest integer less than or equal to x for any real x. For example, $f(1) = 1, f(3/2) = 1, f(7/2) = 3, f(7/3) = 2$. Let

$$g(x) = f(x) + f(x/2) + f(x/3) + \cdots + f(x/(x-1)) + f(x/x).$$

a) Calculate $g(4) - g(3)$ and $g(7) - g(6)$.

b) What is $g(116) - g(115)$?

CC285. Find all values of k so that $x^2+y^2=k^2$ will intersect the circle with equation $(x-5)^2+(y+12)^2=49$ at exactly one point.

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CC281. À l'époque des six premières équipes de la LNH, lors d'une saison particulière, chaque équipe jouait 20 matchs (chaque équipe rencontrait chacune des 5 autres équipes 4 fois). Chaque match se terminait par une victoire, une défaite ou un match nul (il n'y avait aucun jeu en temps supplémentaire). Le tableau suivant présente le classement à la fin de cette saison. Quels sont tous les résultats possibles quant au nombre X de victoires, au nombre Y de défaites et au nombre Z de matchs nuls de l'équipe de Montréal?

Équipe	Victoires	Défaites	Matchs nuls
Toronto	2	12	6
Boston	6	10	4
Detroit	7	12	1
New York	7	9	4
Chicago	11	7	2
Montréal	X	Y	Z

CC282. Calculer la valeur de

$$\left(3^{4/3}-3^{1/3}\right)^3+\left(3^{5/3}-3^{2/3}\right)^3+\left(3^{6/3}-3^{3/3}\right)^3+\cdots+\left(3^{2006/3}-3^{2003/3}\right)^3.$$

CC283. Deux sacs, A et B, contiennent chacun 9 boules. Dans chaque sac, les 9 boules sont numérotées de 1 à 9. On retire au hasard une boule du sac A et une boule du sac B. Soit X la somme des numéros sur les boules qui se trouvent encore dans le sac A et Y la somme des numéros sur les boules qui se trouvent encore dans le sac B. Quelle est la probabilité pour que la différence entre X et Y soit un multiple de 4?

CC284. On définit la fonction f sur l'ensemble des réels comme suit: $f(x)$ est le plus grand entier inférieur ou égal à x. Par exemple, $f(1)=1$, $f(\frac{3}{2})=1$, $f(\frac{7}{2})=3$ et $f(\frac{7}{3})=2$. Soit

$$g(x)=f(x)+f(x/2)+f(x/3)+\cdots+f(x/(x-1))+f(x/x).$$

a) Calculer $g(4)-g(3)$ et $g(7)-g(6)$.

b) Quelle est la valeur de $g(116)-g(115)$?

CC285. Déterminer toutes les valeurs de k pour que le cercle d'équation $x^2+y^2=k^2$ et le cercle d'équation $(x-5)^2+(y+12)^2=49$ se coupent en exactement un point.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(7), p. 291–293.

CC231. If $x^2 + y^2 = 6xy$ with $y > x > 0$, find $\frac{x+y}{x-y}$.

Originally Question 6 of The Seventh W.J. Blundon Contest, 1990.

We received 19 submissions of which 15 were correct and complete. We present 2 solutions.

Solution 1, by Dan Daniel.

We have

$$x^2 + y^2 = 6xy \iff 2x^2 + 2y^2 - 4xy = x^2 + y^2 + 2xy \iff 2(x-y)^2 = (x+y)^2,$$

so

$$\left(\frac{x+y}{x-y}\right)^2 = 2.$$

Since $y > x$, then $\dfrac{x+y}{x-y} = -\sqrt{2}$.

Solution 2, by Titu Zvonaru.

Let $t = \dfrac{y}{x}$. From $x^2 + y^2 = 6xy$ we obtain

$$t^2 - 6t + 1 = 0.$$

Since $t > 1$, we get

$$\frac{x+y}{x-y} = \frac{1+t}{1-t} = \frac{1+3+\sqrt{8}}{1-3-\sqrt{8}} = -\frac{2+\sqrt{2}}{1+\sqrt{2}} = -\sqrt{2}.$$

Editor's Comments. All the wrong submissions reported as a result $\sqrt{2}$ instead of $-\sqrt{2}$. Someone did a mistake when copying the problem (wrote $x > y$ instead of $y > x$), someone forgot that $y > x$, so when you take the square root of $(x-y)^2$ you get a negative number. Konstantine Zelator also considered the general case

$$x^2 + y^2 = kxy, \quad k \text{ is a real number bigger than } 2,$$

and proved that

$$\frac{x+y}{x-y} = -\sqrt{\frac{k+2}{k-2}}.$$

CC232. Seven tests are given and on each test no ties are possible. Each person who is the top scorer on at least one of the tests or who is in the top six on at least four of these tests is given an award, but each person can receive at most one award. Find the maximum number of people who could be given awards if 100 students take these tests.

Originally Team Question 3 of the 1988 Florida Mathematics Olympiad.

We received four correct solutions. We present a combination of all four solutions.

The maximum number of people who could be given awards is 15. There are always 7 top scorers who get an award. The other awards are given to people who were in the top six in at least 4 tests. Altogether 35 people are ranked 2nd, 3rd, 4th, 5th, and 6th. The maximum number of people who could be given awards will be reached if there are as many people who are four times in the top 6 as possible.

$35 = 4 \cdot 8 + 3$, so the maximum number of people who could get awards by being four times in the top 6 is 8. Thus, the maximum number of students that can be given an award is $7 + 8 = 15$; and this works as long as there are at least 15 test-takers, whereas number 100 does not play any special role. A specific set of outcomes with 15 awards can be realized by

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7
1^{st}	A	B	C	D	E	F	G
2^{nd}	H	H	H	H	I	I	I
3^{rd}	I	J	J	J	J	K	K
4^{th}	K	K	L	L	L	L	M
5^{th}	M	M	M	N	N	N	N
6^{th}	O	O	O	O	X	Y	Z

CC233. Let P be a point in the interior of the rectangle $ABCD$. Suppose that $PA = a, PB = b$ and $PC = c$, find, in terms of a, b, c, the length of the line segment PD.

Originally Individual Question 12 (b) of the 1988 Florida Mathematics Olympiad.

We received 13 correct solutions. We present the solution by Titu Zvonaru.

Let P_1 and P_2 be the projections of P onto AB and AD, respectively. By the Pythagorean Theorem,

$$(PP_1)^2+(PP_2)^2 = a^2, (AB-PP_1)^2+(PP_2)^2 = b^2, (AD-PP_2)^2+(AB-PP_1)^2 = c^2.$$

It follows that

$$(PD)^2 = (AD - PP_2)^2 + (AB - PP_1)^2 = a^2 + c^2 - b^2,$$

so that

$$PD = \sqrt{a^2 + c^2 - b^2}.$$

CC234. Find B if
$$x = \frac{\log_{10} 16/3}{\log_{10} B}$$
is the solution to the exponential equation
$$2^{2x+4} + 3^{3x+2} = 4^{x+3}.$$

Originally Individual Question 10 of the 1988 Florida Mathematics Olympiad.

We received 14 correct solutions and one incorrect solution. We present the solution by Kathleen Lewis.

The given equation $2^{2x+4} + 3^{3x+2} = 4^{x+3}$ can be rewritten as
$$3^{3x+2} = 4^{x+3} - 4^{x+2} = 3 \cdot 4^{x+2}.$$
Thus
$$3 \cdot 3^{3x} = 16 \cdot 4^x,$$
so $27^x/4^x = 16/3$. Then
$$x \log_{10}(27/4) = \log_{10}(16/3),$$
so $B = 27/4$.

CC235. Find the area of a regular octagon formed by cutting equal isosceles triangles from the corners of a square with sides of one unit.

Originally Question 6 of The Ninth W.J. Blundon Contest, 1992.

We received 13 correct solutions. We present the solution by Kathleen Lewis.

To end up with a regular octagon with sides of length x, each triangle that is cut off will have legs of length $x/\sqrt{2}$. The original square was a unit square, so
$$1 = x + 2 \cdot x/\sqrt{2} = x(1 + \sqrt{2}),$$
implying that
$$x = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1.$$
Each triangle that was cut off has area $x^2/4$, so the total area removed is $x^2 = 3 - 2\sqrt{2}$. Therefore, the remaining area is
$$1 - (3 - 2\sqrt{2}) = 2\sqrt{2} - 2.$$

THE OLYMPIAD CORNER

No. 355

Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **March 1, 2018**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC341. There are 30 teams in the NBA and every team plays 82 games in the year. Owners of the NBA teams want to divide all teams into Western and Eastern Conferences (not necessarily equally), such that the number of games between teams from different conferences is half of number of all games. Can they do it?

OC342. Consider the second-degree polynomial $P(x) = 4x^2 + 12x - 3015$. Define the sequence of polynomials $P_1(x) = \frac{P(x)}{2016}$ and $P_{n+1}(x) = \frac{P(P_n(x))}{2016}$ for every integer $n \geq 1$.

a) Show that there exists a real number r such that $P_n(r) < 0$ for every positive integer n.

b) For how many integers m does $P_n(m) < 0$ hold for infinitely many positive integers n?

OC343. Determine all pairs of positive integers (a, n) with $a \geq n \geq 2$ for which $(a+1)^n + a - 1$ is a power of 2.

OC344. Find all $a \in \mathbb{R}$ such that there exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying

- $f(1) = 2016$;
- $f(x + y + f(y)) = f(x) + ay \quad \forall x, y \in \mathbb{R}$.

OC345. Let $\triangle ABC$ be an acute triangle, and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$. Lines $\overleftrightarrow{I_B F}$ and $\overleftrightarrow{I_C E}$ meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

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OC341. La NBA compte 30 équipes et chaque équipe joue 82 matchs au cours de la saison régulière. Les propriétaires des équipes de la NBA aimeraient placer les équipes dans la conférence de l'est ou dans la conférence de l'ouest (sans que les conférences aient nécessairement un nombre égal d'équipes), de manière que le nombre de matchs entre des équipes de chaque conférence soit égal à la moitié du nombre total de matchs. Peuvent-ils réussir à le faire?

OC342. On considère le polynôme du second degré $P(x) = 4x^2 + 12x - 3015$. On définit une suite de polynômes comme suit:

$$P_1(x) = \frac{P(x)}{2016} \text{ et } P_{n+1}(x) = \frac{P(P_n(x))}{2016} \text{ pour tout entier } n \ (n \geq 1).$$

a) Démontrer qu'il existe un réel r tel que $P_n(r) < 0$ pour tous entiers n strictement positifs.

b) Combien y a-t-il d'entiers m tels que $P_n(m) < 0$ pour un nombre infini d'entiers n strictement positifs?

OC343. Déterminer tous les couples d'entiers (a, n) $(a \geq n \geq 2)$ pour lesquels la valeur de $(a + 1)^n + a - 1$ est une puissance de 2.

OC344. Déterminer tous les réels a pour lesquels il existe une fonction f $(f : \mathbb{R} \to \mathbb{R})$ qui satisfait à

- $f(1) = 2016$,
- $f(x + y + f(y)) = f(x) + ay \quad \forall x, y \in \mathbb{R}$.

OC345. Soit ABC un triangle acutangle et O le centre du cercle circonscrit au triangle. Soit I_B et I_C les centres respectifs des cercles exinscrits dans les angles B et C. Des points E et Y sont choisis sur le segment AC tels que $\angle ABY = \angle CBY$ et $BE \perp AC$. De même, des points F et Z sont choisis sur le segment AB tels que $\angle ACZ = \angle BCZ$ et $CF \perp AB$. Les droites I_BF et I_CE se coupent en P. Démontrer que les segments PO et YZ sont perpendiculaires.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(5), p. 202–204.

OC281. Find all polynomials $P(x)$ with real coefficients such that

$$P(P(x)) = (x^2 + x + 1) \cdot P(x)$$

where $x \in \mathbb{R}$.

Originally problem 3 from the Junior Level of the 2015 Azerbaijan National Olympiad.

We received 8 correct submissions. We present the solution by Pedro Acosta.

First, observe that the zero polynomial works so throughout assume that P is not the zero polynomial.

Let $d = \deg P$. Then $\deg(P \circ P) = d^2$ and $\deg(P(x) \cdot (x^2 + x + 1)) = d+2$. Thus $d^2 = d + 2 \Rightarrow d = 2$.

Now let α be a root of P. Then

$$P(0) = P(P(\alpha)) = P(\alpha) \cdot (\alpha^2 + \alpha + 1) = 0,$$

so 0 is a root of P. Furthermore, from comparing leading coefficients we see that P must be monic. Thus $P(x) = x^2 - ax$ for some real number a; it suffices to compute all possible values of a.

To do this, note that if ω is a root of $x^2 + x + 1$, then $P(P(\omega)) = 0$. Thus $P(\omega) = 0$ or $P(\omega) = a$. In the former case, we must have $P(x) = x^2 - \omega x$, but this contradicts the fact that the coefficients of the polynomial are real. In the latter case, we then have

$$\omega^2 - a\omega = a \quad \Longrightarrow \quad \omega^2 = a(\omega + 1) = -a\omega^2,$$

i.e. $a = -1$. As a result, $P(x) = x^2 + x$ is the only possible nonzero polynomial P, and one can check that it works.

Editor's Comment: just a gentle reminder to solvers to remember to check the zero polynomial when using polynomial degree arguments.

OC282. Let x, y, z be three nonzero real numbers satisfying $x + y + z = xyz$. Prove that

$$\sum \left(\frac{x^2 - 1}{x}\right)^2 \geq 4.$$

Originally problem 1 of the 2015 Iranian Mathematical Olympiad.

We received 5 correct submissions. We present the solution by Daniel Sitaru.

Let A, B, C be the angles of ΔABC and let $x = \tan A, y = \tan B, z = \tan C$ with sides a, b, c and semi-perimeter S, valid since:

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C \Leftrightarrow x + y + z = xyz.$$

Then,

$$\begin{aligned}\sum\Big(\frac{x^2-1}{x}\Big)^2 \geq 4 &\Leftrightarrow \sum\Big(\frac{\tan^2 A - 1}{\tan A}\Big)^2 \geq 4\\ &\Leftrightarrow \sum(2\cot(2A))^2 \geq 1\\ &\Leftrightarrow \sum \cot^2(2A) \geq 1.\end{aligned}$$

Now, notice that

$$\sqrt{\frac{\sum \cot^2(2A)}{3}} \geq \frac{\sum \cot(2A)}{3}$$

and so

$$\begin{aligned}\sum \cot^2(2A) &\geq \frac{(\sum \cot(2A))^2}{3}\\ &= \frac{1}{3}\Big(\frac{b^2+c^2-a^2}{4S} + \frac{a^2+b^2-c^2}{4S} + \frac{a^2+c^2-b^2}{4S}\Big)^2\\ &= \frac{1}{3}\Big(\frac{a^2+b^2+c^2}{4S}\Big)^2.\end{aligned}$$

It suffices to show this last term above is at least one. However, this follows from the Ionescu-Weitzenbock's inequality which states that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S$$

and hence

$$(a^2 + b^2 + c^2) \geq 48S^2 \Leftrightarrow .\frac{1}{3}\Big(\frac{a^2+b^2+c^2}{4S}\Big)^2 \geq 1$$

OC283. In isosceles $\triangle ABC$, $AB = AC$, I is its incenter, D is a point inside $\triangle ABC$ such that I, B, C, D are concyclic. The line through C parallel to BD meets AD at E. Prove that $CD^2 = BD \cdot CE$.

Originally problem 2 from Part B of the 2015 China Second Round Olympiad.

We received 2 correct submissions. We present the solution by Andrea Fanchini.

We use barycentric coordinates and the usual Conway's notations with reference to triangle ABC. First of all we have

$$b = c, \qquad S_B = S_C = \frac{a^2}{2}, \qquad I(a : b : b)$$

Equation of a generic circle is

$$a^2yz + b^2zx + c^2xy - (x + y + z)(px + qy + rz) = 0.$$

If this circle passes through B, C, I, we obtain the three conditions

$$q = 0, \qquad r = 0, \qquad p = b^2,$$

so this circle has equation

$$a^2yz + b^2zx + c^2xy - (x + y + z)b^2x = 0 \quad \Rightarrow \quad a^2yz - b^2x^2 = 0.$$

Then a generic point D on this circle and inside $\triangle ABC$ has coordinates $D(a\sqrt{t} : b : bt)$ where t is a parameter.

Lines AD and BD have equations

$$AD : ty - z = 0, \qquad BD : btx - a\sqrt{t}z = 0.$$

The line through C parallel to BD is

$$CBD_\infty : (a\sqrt{t} + bt)x + a\sqrt{t}y = 0,$$

therefore point E is the intersection between AD and CBD_∞

$$E = AD \cap CBD_\infty = (a\sqrt{t} : -a\sqrt{t} - bt : -at\sqrt{t} - bt^2).$$

Now we can calculate the distances CD, BD, CE that are

$$CD^2 = \frac{a^2b}{a\sqrt{t} + bt + b}, \qquad BD = \sqrt{\frac{a^2bt}{a\sqrt{t} + bt + b}}, \qquad CE = \sqrt{\frac{a^2b}{t(a\sqrt{t} + bt + b)}}$$

and we are done.

OC284. A positive integer n is given. If there exists sets $F_1, F_2, \ldots, F_m$ satisfying the following conditions, prove that $m \leq n$.

1. For all $1 \leq i \leq m$, $F_i \subseteq \{1, 2, \cdots, n\}$
2. For all $1 \leq i < j \leq m$, $\min(|F_i - F_j|, |F_j - F_i|) = 1$

Originally problem 3 from day 2 of the 2015 Korea National Olympiad.

We present the solution by Steven Chow. There were no other submissions.

It shall be proved using mathematical induction on n.

If $n = 1$, then $m \leq 1 \leq n$. (From condition 2, for all $1 \leq j \leq m$, $F_j \neq \emptyset$.)

If $n = 2$, then $m \leq 2 = n$: $\min\left(|\{1\} \backslash \{1, 2\}|, |\{1, 2\} \backslash \{1\}|\right) = \min(0, 1) = 0 \neq 1$.

Assume for the sake of contradiction that $m \geq k + 1$. For all $1 \leq j \leq m$, $1 \leq |F_j| \leq k$, so there exists $1 \leq a < b \leq m$ such that $|F_a| = |F_b|$. Since we have $\min\left(|F_a \backslash F_b|, |F_b \backslash F_a|\right) = 1$, therefore $|F_a \backslash (F_a \cap F_b)| = |F_b \backslash (F_a \cap F_b)| = 1$, so there exists $1 \leq r, s \leq k$ such that $r \in F_a \backslash (F_a \cap F_b)$ and $s \in F_b \backslash (F_a \cap F_b)$.

If there exists $1 \leq c \leq m$ such that $c \neq a, b$ and $r \in F_c$, $s \notin F_c$, then since $\min(|F_b \backslash F_c|, |F_c \backslash F_b|) = 1$, it follows that $F_b \backslash F_c = \{s\}$ and $F_c \backslash F_b = \{r\}$, so $F_b \cap F_c = F_a \cap F_b$ and $F_c = F_a$ which is a contradiction. Therefore there does not exist $1 \leq c \leq m$ such that $c \neq a, b$ and $r \in F_c$, $s \notin F_c$, and similarly there does not exist $1 \leq c \leq m$ such that $c \neq a, b$ and $r \notin F_c$, $s \in F_c$.

Therefore for all integers $1 \leq j \leq m$ such that $j \neq a, b$, either $r, s \notin F_j$ or $r, s \in F_j$, so the sets F_j not containing r or s satisfy conditions 1 and 2.

From the induction hypothesis, we have $m \leq 2 + (k-2) = k$. A contradiction.

Therefore $m \leq k$, and if $n = k$, then the statement is true. Therefore $m \leq n$.

OC285. Show that from a set of 11 square integers one can select six numbers $a^2, b^2, c^2, d^2, e^2, f^2$ such that $a^2 + b^2 + c^2 \equiv d^2 + e^2 + f^2 \pmod{12}$.

Originally problem 6 of the 2015 India National Olympiad.

We received 3 correct submissions. We present the solution by Somasundaram Muralidharan.

When the square of an integer is divided by 12, the possible remainders are $0, 1, 4, 9$. Out of the 11 square numbers, if there are 6 or more numbers that leave the same remainder r when divided by 12, then clearly we can select 2 sets of three numbers from them such that their sums leave the same remainder $3r$ when divided by 12. Suppose that for any remainder r there are no more than 5 numbers n such that n^2 leaves the remainder r when divided by 12. Firstly, suppose that there is a remainder r_1 for which there are 4 numbers n such that n^2 leaves a remainder r_1 when divided by 12. Out of the remaining 7 numbers, there is at least one remainder $r_2 \neq r_1$ such that there are at least two n' whose squares leave remainder r_2. Now we can choose a, b, d, e whose squares leave remainder r_1 and c, f whose squares leave remainder r_2. Then $a^2 + b^2 + c^2 = (2r_1 + r_2) \mod 12 = d^2 + e^2 + f^2$. We are now left with the case in which the split of the numbers is $3, 3, 3, 2$ for the remainders. In this case we can take numbers such that

$$a^2 = 0 \mod 12, \quad b^2 = 9 \mod 12, \quad c^2 = 9 \mod 12,$$
$$d^2 = 1 \mod 12, \quad e^2 = 1 \mod 12, \quad f^2 = 4 \mod 12.$$

Then we have

$$a^2 + b^2 + c^2 = 6 \mod 12,$$
$$d^2 + e^2 + f^2 = 6 \mod 12.$$

This completes the proof.

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Editor's Comments. Congratulations to Steven Chow who solved all Olympiad Corner problems correctly.

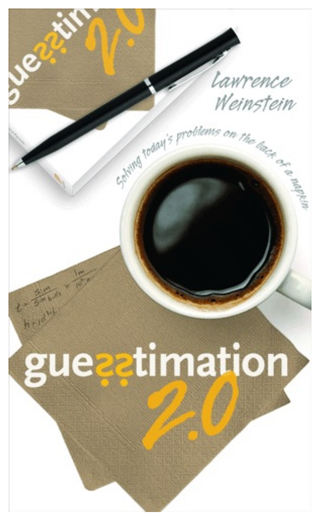
BOOK REVIEWS

Robert Bilinski

Guesstimation 2.0 by L. Weinstein
ISBN 978-0-691-15080-2, 359 pages
Published by Princeton University Press, 2012

Reviewed by ***Robert Bilinski****, Collège Montmorency.*

Lawrence Weinstein is a physics professor at Old Dominion University, Norfolk, Virginia. He works at a particle accelerator to discover how matter is composed. Based on his personal experience as a teacher using a problem based approach, he has already published a first Guesstimation book in 2008. With a self-explanatory title, the readers of ***Crux*** will have guessed (pun intended) that the aim of the book is coming up with rough solutions that are part guess and part estimation, but doing so in as rigorous a way as possible.



Problem solving is the name of the game, but problem solving can mean a lot of things to a lot of people. Lawrence Weinsten's take on it as a physics professor is that you can solve a lot of real-life situations with the use of mathematics. The thesis underpinning the technique of guesstimation is that the situation in which the problem arises might not even need to be fully mathematicised to be solved by mathematics. To give an example of the situations covered, we can look at the table of contents: 1) How to solve problems, 2) General questionss, 3) Recycling: what really matters?, 4) The five senses, 5) Energy and work, 6) Energy and transportation, 7) Heavenly bodies, 8) Materials, 9) Radiation, and 2 appendices. We can see from the titles that the first 6 chapters are aimed at laypersons and the last 3 are targeting a more scientific audience, but things aren't that simple. A lot of the questions have a humorous bent to them. Most problems also seem to have a second purpose, which is to help the reader get a better sense of our environments. Each chapter contains between 8 and 16 situations (I'll let you guesstimate how many are in total in the book). The first appendix aims to develop the number sense in the reader, while the second one gives a list of common things and their respective orders of magnitude, which are used to guesstimate various quantities in the book, but also to give concrete tools to permit the readers to practice guesstimating.

The approach taken is to create reasonable solutions and give an interval in which one has a good chance of finding the solution. The whole gist of the book is that these rough approximations are usually enough in a concrete and real life situation to function. The approach is not statistical so one cannot talk about quantifying risk of making an error. A fact I particularly enjoyed was that the whole technique is based on the geometric mean, which in my opinion is under-taught and under-used.

To get a better feel for the book, here are some questions posed and then answered in the book :

- How much does a trillion dollars weigh?
- Which has more mass, the air or the brains in a movie theatre?
- What is the maximum amount of light that our eyes can tolerate, even briefly?
- How fast would the sun rotate if it collapsed into a neutron star?
- What the maximal height a mountain can have on earth? (rephrased)

Maybe my favourite ones are the problems on recycling where the author compares different ways of proceeding or answers indirectly under what conditions recycling is worthwhile.

The readers of ***Crux*** , all problem solvers themselves, will appreciate the medley of situations where their art is applied in this book, as well as the tricks used to solve them. I wonder how many will appreciate the messiness of the solutions. A lot of our younger readers will surely jump at the chance of seeing mathematics applied in the concrete world, especially with the irreverent tone given to many problems posed. Good reading!

FOCUS ON...

No. 27

Michel Bataille

Some relations in the triangle (I)

Introduction

Relations between the elements of a triangle intervene in ***Crux*** problems – and solutions – quite often, and that's an understatement! Every solver wanting to establish an identity or an inequality involving those elements should have a number of these relations in her/his toolbox: the Laws of Sines and Cosines of course, but also classical relations such as $a = b\cos C + c\cos B$ or $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

However, even the most common of these relations form a vast subject; for instance, the compendium of them in **[1]** spreads over some twenty pages! The modest goal of this number and the next one is to present a selection of less familiar relations chosen because of their aesthetic qualities and/or their applications to problems.

Here and in what follows, we use the standard notations as they can be found in **[2]** (where α, β, γ are preferred to A, B, C for the angles of the triangle, though).

About the differences of angles $B - C, C - A$ and $A - B$

We begin with the surprising relation

$$a\cos(B-C) + b\cos(C-A) + c\cos(A-B) = 2(a\cos A + b\cos B + c\cos C). \quad (1)$$

The proof is easy: we have that $a\sin C = c\sin A$ (from the Law of Sines) and $b = a\cos C + c\cos A$, therefore

$$\begin{aligned} a\cos(B-C) - b\cos B &= a\cos B\cos C + a\sin B\sin C - b\cos B \\ &= \cos B(a\cos C - b) + c\sin A\sin B \\ &= -c\cos A\cos B + c\sin A\sin B \\ &= -c\cos(A+B) \\ &= c\cos C. \end{aligned}$$

Hence

$$a\cos(B-C) = b\cos B + c\cos C.$$

With similar results for $b\cos(C-A)$ and $c\cos(A-B)$, the desired equality is deduced at once.

With the help of the Law of Cosines, (1) leads to

$$a\cos(B-C) + b\cos(C-A) + c\cos(A-B) = \frac{2(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4}{abc} \quad (2)$$

so that

$$a\cos(B-C)+b\cos(C-A)+c\cos(A-B)=\frac{16F^2}{4RF}=\frac{4F}{R}=\frac{abc}{R^2}=\frac{4rs}{R}.$$

Here are two applications. First, the latter shows that $\frac{4rs}{R}\leq a+b+c=2s$ and we obtain Euler's inequality $R\geq 2r$ in a rather oblique way!

Second, (1) yields a neat solution to problem **2546** [2000 : 237 ; 2001 : 343]

Prove that triangle ABC is equilateral if and only if

$$a\cos(B-C)+b\cos(C-A)+c\cos(A-B)=\frac{a^4+b^4+c^4}{abc},$$

for (2) shows that the given relation is equivalent to

$$2(a^4+b^4+c^4)=2(a^2b^2+b^2c^2+c^2a^2),$$

that is, to

$$(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2=0,$$

which clearly holds if and only if $a=b=c$.

In the same vein, we will prove the following beautiful formula

$$a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)=3abc. \qquad (3)$$

To this aim, we start with expressions of the area F that deserve to be better known:

$$4F=a^2\sin 2B+b^2\sin 2A=b^2\sin 2C+c^2\sin 2B=c^2\sin 2A+a^2\sin 2C. \qquad (4)$$

This follows, for example, from

$$a^2\sin 2B+b^2\sin 2A=2a^2\sin B\cos B+2ab\sin B\cos A=2ac\sin B=4F.$$

Formula (3) is deduced smoothly once we have noticed that

$$\begin{aligned}a^3\cos(B-C)&=a^2\cdot 2R\sin A\cos(B-C)\\&=2Ra^2\sin(B+C)\cos(B-C)\\&=Ra^2(\sin 2B+\sin 2C).\end{aligned}$$

Then, using (4),

$$\begin{aligned}&\sum_{\text{cyclic}}a^3\cos(B-C)\\&\quad=R(a^2\sin 2B+a^2\sin 2C+b^2\sin 2C+b^2\sin 2A+c^2\sin 2A+c^2\sin 2B)\\&\quad=R(4F+4F+4F)\\&\quad=12RF\\&\quad=3abc.\end{aligned}$$

About the cosines of $\dfrac{B-C}{2}$, $\dfrac{C-A}{2}$ and $\dfrac{A-B}{2}$

We will consider two expressions of $\cos\left(\frac{B-C}{2}\right)$ and apply them to past problems.

The first one is readily obtained:

$$\cos\left(\frac{B-C}{2}\right) = \frac{b+c}{a}\cdot\sin\frac{A}{2}. \tag{5}$$

Indeed, because $\cos\frac{A}{2} = \sin(\frac{B+C}{2})$, we have

$$\frac{\cos(\frac{B-C}{2})}{\sin\frac{A}{2}} = \frac{2\sin(\frac{B+C}{2})\cos(\frac{B-C}{2})}{2\cos\frac{A}{2}\sin\frac{A}{2}} = \frac{\sin B+\sin C}{\sin A} = \frac{b+c}{a}.$$

Relation (5) and similar relations provide a quick and easy solution to problem **2717** [2002 : 112; 2003 : 119], which states

For any triangle ABC, prove that

$$8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right).$$

This immediately follows from (5):

$$\begin{aligned}\frac{\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} &= \left(\frac{b}{a}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{a}{b}\right)\left(\frac{a}{c}+\frac{b}{c}\right)\\ &= 2+\left(\frac{b}{a}+\frac{a}{b}\right)+\left(\frac{c}{b}+\frac{b}{c}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)\\ &\ge 8,\end{aligned}$$

since $x+\frac{1}{x}\ge 2$ for all positive real numbers x.

Note that the inequality rewrites as

$$\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right) \ge \frac{2r}{R}. \tag{6}$$

From (6) we easily derive the related inequality

$$\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{C-A}{2}\right) \ge 1+\frac{4r}{R}. \tag{7}$$

Proof. Since

$$\cos\left(\frac{A-B}{2}\right) \ge \cos^2\left(\frac{A-B}{2}\right) = \frac{1}{2}+\frac{1}{2}\cos(A-B)$$

it is sufficient to show that

$$\cos(A-B)+\cos(B-C)+\cos(C-A) \ge \frac{8r}{R}-1.$$

But, when $x+y+z=0$, we have

$$\cos x+\cos y+\cos z=4\cos\frac{x}{2}\cos\frac{y}{2}\cos\frac{z}{2}-1,$$

hence the latter inequality readily transforms into (6) and therefore is true.

It is interesting to notice that this is a variant of proof of an old ***Crux*** inequality, namely

$$\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{C-A}{2}\right)\geq 4(\cos A+\cos B+\cos C)-3,$$

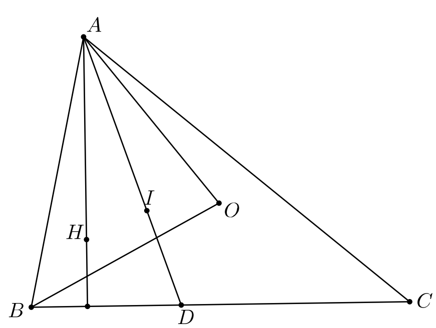
proposed in problem **696** [1981 : 302 ; 1982 : 316] (taking into account the known formula $\cos A+\cos B+\cos C=1+\frac{r}{R}$).

The following additional formula involves the distance IA from the incenter I to the vertex A:

$$\cos\left(\frac{B-C}{2}\right)=\frac{IA}{2R}+\frac{r}{IA}.$$

To prove this relation, we introduce the circumcentre O of ABC and observe that in the case when $B\geq C$, we have $C\leq 90^\circ$ and

$$\angle BAO=\angle OBA=\frac{180^\circ-\angle AOB}{2}=90^\circ-C.$$



In consequence,

$$\angle IAO=\angle BAO-\angle BAI=90^\circ-C-\frac{A}{2}=\frac{B-C}{2}.$$

If $C>B$, we obtain $\angle IAO=\frac{C-B}{2}$ and in either case

$$\cos\left(\frac{B-C}{2}\right)=\frac{IA^2+R^2-IO^2}{2IA\cdot R}=\frac{IA}{2R}+\frac{r}{IA}$$

since $IO^2=R^2-2rR$ (Euler's formula).

Note that an application of the arithmetic-geometric mean inequality yields

$$\cos\left(\frac{B-C}{2}\right)\geq 2\sqrt{\frac{IA}{2R}\cdot\frac{r}{IA}}=2\sqrt{\frac{r}{2R}}$$

or

$$\cos^2\left(\frac{B-C}{2}\right) \geq \frac{2r}{R},$$

the inequality to be proved in problem **2382** [1998 : 425 ; 1999 : 440].

Our second part will offer relations involving the altitudes, exradii, *etc*. By way of transition, let us remark that if H is the orthocentre of ΔABC, then $\angle IAO = \angle IAH$ (recall that the line AO and the altitude from vertex A are symmetric in the angle bisector of $\angle BAC$). We deduce that $\cos\left(\frac{B-C}{2}\right) = \frac{h_a}{w_a}$ where, following the notations of **[2]**, $w_a = AD$ is the length of the angle bisector of $\angle BAC$ (see figure above). Therefore, relations (6) and (7) yield

$$\frac{h_a h_b h_c}{w_a w_b w_c} \geq \frac{2r}{R} \qquad \text{and} \qquad \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \geq 1 + \frac{4r}{R}.$$

Exercises

1.

a) Establish the formula $\sin 2A + \sin 2B + \sin 2C = \dfrac{abc}{2R^3}$ and deduce an expression of $a^2 \sin 2A + b^2 \sin 2B + c^2 \sin 2C$.

b) Prove that

$$a^3 \cos A + b^3 \cos B + c^3 \cos C = \frac{abc}{2R^2} \cdot (a^2 + b^2 + c^2 - 6R^2)$$

and

$$a\cos^3 A + b\cos^3 B + c\cos^3 C = \frac{abc}{8R^4} \cdot (10R^2 - (a^2 + b^2 + c^2)).$$

c) From the latter, deduce that if ΔABC is not obtuse then

$$a\cos^3 A + b\cos^3 B + c\cos^3 C \leq \frac{abc}{4R^2}.$$

(the inequality to be proved in problem **3167** [2006 : 395,397 ; 2007 : 374].)

2. Prove the inequality $\sum_{\text{cyclic}} a\cos\frac{B-C}{2} \geq s\left(1 + \frac{2r}{R}\right)$ (use (5) or give a look at problem **696**) and deduce that

$$\frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c} \geq \frac{1}{2r} + \frac{1}{R}.$$

References

[1] T. Lalesco, *La géométrie du triangle*, J. Gabay, 2003, p. 101-120

[2] O. Bottema *et al.*, *Geometric Inequalities*, Wolters-Noordhoff, 1968, p. 9-10

A "probabilistic" method for proving inequalities

Daniel Sitaru and Claudia Nănuţi

In this paper we solve a class of inequalities using an identity familiar from probability theory and classical mechanics.

In the year 2000, Fuhua Wei and Shan - He Wu from the Department of Mathematics and Computer Science, Longyan University, Longyan, Fujian 364012, P.R. China published the article: "Several proofs and generalisations of a fractional inequality with constraints." In this article, they give ten different proofs for the 2nd problem of the 36th IMO, held at Toronto (Canada) in 1995.

In proof 5, the authors used a method based on a key random variable to prove that if a, b, c are positive real numbers with $abc = 1$ then:

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

Proof. We make the substitutions $x := bc$, $y := ca$, $z := ab$, and $s := x + y + z$.

Then:

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(a+b)} = \frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y} = \frac{x^2}{s-x}+\frac{y^2}{s-y}+\frac{z^2}{s-z}.$$

We consider the random variable ξ defined as follows:

$$\xi = \begin{cases} \frac{x}{s-x} : & (p = \frac{s-x}{2s}), \\ \frac{y}{s-y} : & (p = \frac{s-y}{2s}), \\ \frac{z}{s-z} : & (p = \frac{s-z}{2s}). \end{cases}$$

It follows that

$$E(\xi) = \frac{x}{s-x} \cdot \frac{s-x}{2s} + \frac{y}{s-y} \cdot \frac{s-y}{2s} + \frac{z}{s-z} \cdot \frac{s-z}{2s} = \frac{x+y+z}{2s} = \frac{1}{2}$$

and also

$$\begin{aligned} E(\xi^2) &= \Big(\frac{x}{s-x}\Big)^2 \cdot \frac{s-x}{2s} + \Big(\frac{y}{s-y}\Big)^2 \cdot \frac{s-y}{2s} + \Big(\frac{z}{s-z}\Big)^2 \cdot \frac{s-z}{2s} \\ &= \frac{1}{2s}\Big(\frac{x^2}{s-x} + \frac{y^2}{s-y} + \frac{z^2}{s-z}\Big). \end{aligned}$$

Now, the variance of ξ is given by $V(\xi) = E(\xi^2) - (E(\xi))^2$. This is always nonnegative, and positive unless ξ can take only one value (in which case $x = y = z$ and $a = b = c$.) We thus have

$$\frac{1}{2s}\Big(\frac{x^2}{s-x} + \frac{y^2}{s-y} + \frac{z^2}{s-z}\Big) \geq \frac{1}{4}$$

and so

$$\frac{x^2}{s-x}+\frac{y^2}{s-y}+\frac{z^2}{s-z}\geq\frac{1}{2}s=\frac{1}{2}(x+y+z)\overbrace{\geq}^{AM-GM}\frac{3}{2}\sqrt[3]{xyz}=\frac{3}{2}.$$

Hence

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}$$

and this is strict unless $a=b=c$. □

The method of proof used here is based on the positivity of variance:

$$E(\xi^2)-(E(\xi))^2=V(\xi)=E((\xi-E(\xi))^2)\geq 0\ ,$$

whence

$$E(\xi^2)\geq(E(\xi))^2\ .$$

It can be applied to other problems as well. The technique is to construct a random variable such that its variance is the quantity, or difference, that we wish to show positive. (Readers familiar with classical mechanics may prefer to consider this in terms of the parallel axis theorem for moments of inertia - a "mechanical" method of proof?)

Example 1. Prove that if $x, y, z > 0$ then:

$$\sqrt{\frac{x}{y}}+2\sqrt{\frac{y}{z}}+3\sqrt{\frac{z}{x}}\leq\sqrt{6\Big(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\Big)}$$

Solution. Define a random variable

$$\xi=\begin{cases}\sqrt{\frac{x}{y}}: & (p=\frac{1}{6}),\\ \sqrt{\frac{y}{z}}: & (p=\frac{2}{6}),\\ \sqrt{\frac{z}{x}}: & (p=\frac{3}{6}),\end{cases}\qquad\text{then}\quad \xi^2=\begin{cases}\frac{x}{y}: & (p=\frac{1}{6}),\\ \frac{y}{z}: & (p=\frac{2}{6}),\\ \frac{z}{x}: & (p=\frac{3}{6}).\end{cases}$$

It follows that

$$E(\xi)=\frac{1}{6}\sqrt{\frac{x}{y}}+\frac{2}{6}\sqrt{\frac{y}{z}}+\frac{3}{6}\sqrt{\frac{z}{x}}\quad\text{and}\quad E(\xi^2)=\frac{1}{6}\Big(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\Big)\ .$$

As

$$E(\xi^2)\geq(E(\xi))^2,$$

we have

$$\frac{1}{6}\Big(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\Big)\geq\Big[\frac{1}{6}\Big(\sqrt{\frac{x}{y}}+2\sqrt{\frac{y}{z}}+3\sqrt{\frac{z}{x}}\Big)\Big]^2,$$

$$\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\geq\frac{1}{6}\Big(\sqrt{\frac{x}{y}}+2\sqrt{\frac{y}{z}}+3\sqrt{\frac{z}{x}}\Big)^2,$$

$$\sqrt{\frac{x}{y}}+2\sqrt{\frac{y}{z}}+3\sqrt{\frac{z}{x}}\leq\sqrt{6\Big(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\Big)}$$

and, again, equality holds only for $x = y = z$. □

Example 2. Prove that if $a, b, c > 0$ then:

$$\sqrt{\frac{a}{b+c}} + 2\sqrt{\frac{b}{c+a}} + 4\sqrt{\frac{c}{a+b}} \leq \sqrt{7\Big(\frac{a}{b+c} + \frac{2b}{c+a} + \frac{4c}{a+b}\Big)}$$

Solution. Define a random variable

$$\xi = \begin{cases} \sqrt{\frac{a}{b+c}} : & (p = \frac{1}{7}), \\ \sqrt{\frac{b}{c+a}} : & (p = \frac{2}{7}), \\ \sqrt{\frac{c}{a+b}} : & (p = \frac{4}{7}). \end{cases}$$

As before we get

$$E(\xi) = \frac{1}{7}\Big(\sqrt{\frac{a}{b+c}} + 2\sqrt{\frac{b}{c+a}} + 4\sqrt{\frac{c}{a+b}}\Big) \quad \text{and} \quad E(\xi^2) = \frac{1}{7}\Big(\frac{a}{b+c} + \frac{2b}{c+a} + \frac{4c}{a+b}\Big),$$

and the inequality

$$\frac{1}{7}\Big(\frac{a}{b+c} + \frac{2b}{c+a} + \frac{4c}{a+b}\Big) \geq \frac{1}{49}\Big(\sqrt{\frac{a}{b+c}} + 2\sqrt{\frac{b}{c+a}} + 4\sqrt{\frac{c}{a+b}}\Big)^2.$$

Therefore

$$\frac{1}{\sqrt{7}} \cdot \sqrt{\frac{a}{b+c} + \frac{2b}{c+a} + \frac{4c}{a+b}} \geq \frac{1}{7}\Big(\sqrt{\frac{a}{b+c}} + 2\sqrt{\frac{b}{c+a}} + 4\sqrt{\frac{c}{a+b}}\Big),$$

and

$$\sqrt{\frac{a}{b+c}} + 2\sqrt{\frac{b}{c+a}} + 4\sqrt{\frac{c}{a+b}} \leq \sqrt{7\Big(\frac{a}{b+c} + \frac{2b}{c+a} + \frac{4c}{a+b}\Big)},$$

with equality only for $a = b = c$. □

Application 3. Prove that in any triangle ABC the following relationship holds for the medians m_a, m_b, m_c and altitudes h_a, h_b, h_c:

$$3\sqrt{\frac{m_a}{m_b} + \frac{2m_b}{m_c} + \frac{6m_c}{m_a}} \geq \sqrt{\frac{h_a}{m_b}} + 2\sqrt{\frac{h_b}{m_c}} + 6\sqrt{\frac{h_c}{m_a}}$$

Solution. Let be the probability distribution sequence of random variable ξ below: Define a random variable

$$\xi = \begin{cases} \sqrt{\frac{m_a}{m_b}} : & (p = \frac{1}{9}) \\ \sqrt{\frac{m_b}{m_c}} : & (p = \frac{2}{9}) \\ \sqrt{\frac{m_c}{m_a}} : & (p = \frac{6}{9}). \end{cases}$$

It follows that

$$E(\xi) = \frac{1}{9}\Big(\sqrt{\frac{m_a}{m_b}} + 2\sqrt{\frac{m_b}{m_c}} + 6\sqrt{\frac{m_c}{m_a}}\Big) \quad \text{and} \quad E(\xi^2) = \frac{1}{9}\Big(\frac{m_a}{m_b} + \frac{2m_b}{m_c} + \frac{6m_c}{m_a}\Big)\ ,$$

and, $m_a \geq h_a$, $m_b \geq h_b$, and $m_c \geq h_c$, we have

$$\begin{aligned} \frac{1}{9}\Big(\frac{m_a}{m_b} + \frac{2m_b}{m_c} + \frac{6m_c}{m_a}\Big) &\geq \frac{1}{81}\Big(\sqrt{\frac{m_a}{m_b}} + 2\sqrt{\frac{m_b}{m_c}} + 6\sqrt{\frac{m_c}{m_a}}\Big)^2 \\ &\geq \frac{1}{81}\Big(\sqrt{\frac{h_a}{m_b}} + 2\sqrt{\frac{h_b}{m_c}} + 6\sqrt{\frac{h_c}{m_a}}\Big)^2\ , \end{aligned}$$

whence

$$9\Big(\frac{m_a}{m_b} + \frac{2m_b}{m_c} + \frac{6m_c}{m_a}\Big) \geq \Big(\sqrt{\frac{h_a}{m_b}} + 2\sqrt{\frac{h_b}{m_c}} + 6\sqrt{\frac{h_c}{m_a}}\Big)^2$$

and

$$3\sqrt{\frac{m_a}{m_b} + \frac{2m_b}{m_c} + \frac{6m_c}{m_a}} \geq \sqrt{\frac{h_a}{m_b}} + 2\sqrt{\frac{h_b}{m_c}} + 6\sqrt{\frac{h_c}{m_a}},$$

which completes the solution. □

Of course, applying this process in reverse is an intriguing way to invent new inequalities!

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[1] Shan - He Wu, Mihaly Bencze, *Selected problems and theorems of analytic inequalities.* Studis Publishing House, Iaşi, Romania, 2012.

[2] Daniel Sitaru, *Math Phenomenon.* Paralela 45 Publishing House, Piteşti, Romania, 2016.

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PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **March 1, 2018**.

The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

An asterisk ($\star$) *after a number indicates that a problem was proposed without a solution.*

4261. *Proposed by Margarita Maksakova.*

Consider the chess board. A baron can move only on the black squares and in one move he can go from one black square to any of the diagonally adjacent black squares. What is the smallest number of moves he needs to go to every black square?

4262. *Proposed by Prithwijit De.*

Let $a_1, a_2, \ldots, a_n$ be positive integers and suppose $\sum_{k=1}^{n} a_k = S$. Find the smallest positive value of c such that the equation

$$\sum_{k=1}^{n} \frac{a_k x^k}{1+x^{2k}} = c$$

has a unique real solution.

4263. *Proposed by Michel Bataille.*

Let ABC be a triangle. Let Γ, with centre O and radius R, be the circumcircle of ABC and γ, with centre $I \neq O$ and radius r, be the incircle of ABC. Let D, E, F be the orthogonal projections of the inverse of I in Γ onto BC, CA, AB, respectively. Express the circumradius of ΔDEF as a function of R and r.

4264. *Proposed by Dorin Marghidanu and Leonard Giugiuc.*

Let (a_n) and (b_n) be two sequences such that $a_0, b_0 > 0$ and

$$a_{n+1} = a_n + \frac{1}{2b_n} \quad \text{and} \quad b_{n+1} = b_n + \frac{1}{2a_n}$$

for all $n \geq 0$. Prove that

$$\max(a_{2017}, b_{2017}) > 44.$$

4265. *Proposed by Daniel Sitaru.*

Consider real numbers $a, b, c \in (0, 1)$ such that $a + b + c = 1$. Show that

$$\frac{4}{\pi}(\arctan a + \arctan b + \arctan c) > \frac{1}{2 - (ab + bc + ca)}.$$

4266. *Proposed by Marius Stănean.*

Let ABC be a triangle with orthocenter H. Let HM be the median and HS be the symmedian in triangle BHC. Denote by P the orthogonal projection of A onto HS. Prove that the circumcircle of triangle MPS is tangent to the circumcircle of triangle ABC.

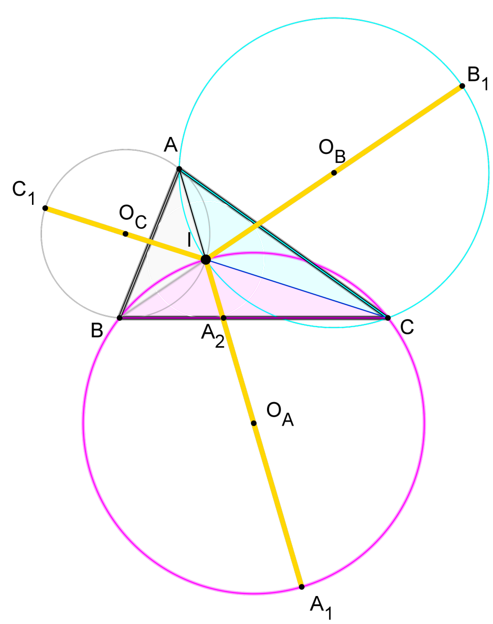
4267. *Proposed by Leonard Giugiuc.*

Let a, b, c and d be real numbers such that $0 < a, b, c \leq 1$ and $abcd = 1$. Prove that

$$5(a + b + c + d) + \frac{4}{abc + abd + acd + bcd} \geq 21.$$

4268. *Proposed by Mihaela Berindeanu.*

Let I be the incenter of the acute triangle ABC, and let the triangle's internal angle bisectors intersect the circles IBC, ICA, and IAB again at A_1, B_1, and C_1, respectively. Show that $\overrightarrow{IA_1} + \overrightarrow{IB_1} + \overrightarrow{IC_1} = \overrightarrow{0}$ if and only if $\triangle\, ABC$ is equilateral.



4269. *Proposed by Hung Nguyen Viet.*

Let $x_1, x_2, \ldots, x_n$ be real numbers such that

$$\sin x_1 \cos x_2 + \sin x_2 \cos x_3 + \cdots + \sin x_n \cos x_1 = \frac{n}{2}.$$

Prove that

$$\cos 2x_1 + \cos 2x_2 + \cdots + \cos 2x_n = 0.$$

4270. *Proposed by Leonard Giugiuc.*

Let k and t be real numbers with $k \in (0,1)$ and $t \in [\frac{\pi}{4}, \frac{\pi}{2}]$. Prove that

$$\int_0^t \frac{\cos x}{x^k} dx \geq \int_0^t \frac{\sin x}{x^k} dx.$$

. . . .

4261. *Proposé par Margarita Maksakova.*

Soit un échiquier. Bernadette y déplace un jeton, allant d'un carré noir à un de ses carrés noirs diagonalement adjacents. Quel est le plus petit nombre de tels déplacements qui permettra de visiter tous les carrés noirs?

4262. *Proposé par Prithwijit De.*

Soient $a_1, a_2, \ldots, a_n$ des entiers positifs tels que $\sum_{k=1}^{n} a_k = S$. Déterminer la plus petite valeur positive de c telle que l'équation

$$\sum_{k=1}^{n} \frac{a_k x^k}{1 + x^{2k}} = c$$

possède une solution réelle unique.

4263. *Proposé par Michel Bataille.*

Soit ABC un triangle avec cercle circonscrit Γ de centre O et rayon R, puis cercle inscrit γ de centre $I \neq O$ et rayon r. Soient D, E et F les projections orthogonales de l'inverse de I dans Γ vers BC, CA et AB respectivement. Exprimer le rayon du cercle circonscrit de ΔDEF en termes de R et r.

4264. *Proposé par Dorin Marghidanu et Leonard Giugiuc.*

Soient (a_n) et (b_n) deux suites telles que $a_0, b_0 > 0$ puis

$$a_{n+1} = a_n + \frac{1}{2b_n} \quad \text{et} \quad b_{n+1} = b_n + \frac{1}{2a_n}$$

pour tout $n \geq 0$. Démontrer que

$$\max(a_{2017}, b_{2017}) > 44.$$

4265. *Proposé par Daniel Sitaru.*

Soient des nombres réels $a, b, c \in (0, 1)$ tels que $a + b + c = 1$. Démontrer que

$$\frac{4}{\pi}(\arctan a + \arctan b + \arctan c) > \frac{1}{2 - (ab + bc + ca)}.$$

4266. *Proposé par Marius Stănean.*

Soit ABC un triangle avec orthocentre H. Soient HM la médiane et HS la symédiane dans le triangle BHC. Dénoter par P la projection orthogonale de A vers HS. Démontrer que le cercle circonscrit du triangle MPS est tangent au cercle circonscrit du triangle ABC.

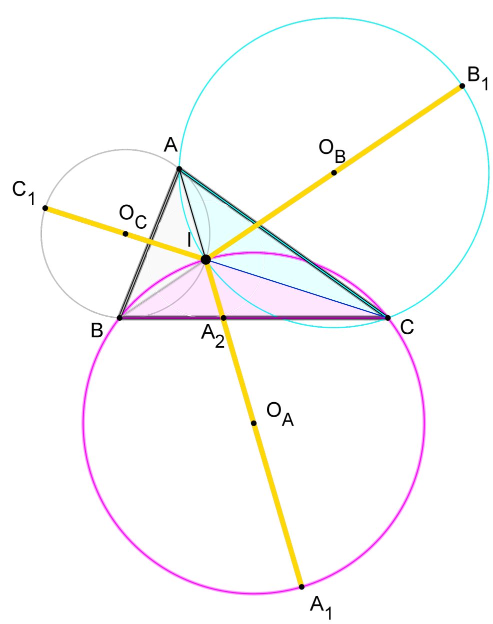
4267. *Proposé par Leonard Giugiuc.*

Soient a, b, c et d des nombres réels tels que $0 < a, b, c \leq 1$ and $abcd = 1$. Démontrer que

$$5(a + b + c + d) + \frac{4}{abc + abd + acd + bcd} \geq 21.$$

4268. *Proposé par Mihaela Berindeanu.*

Soit I le centre du cercle inscrit du triangle aigu ABC et supposer que ses bissectrices internes d'angles intersectent les cercles IBC, ICA et IAB une seconde fois en A_1, B_1 et C_1 respectivement. Démontrer que $\overrightarrow{IA_1} + \overrightarrow{IB_1} + \overrightarrow{IC_1} = \overrightarrow{0}$ si et seulement si $\triangle\, ABC$ est équilatéral.



4269. *Proposé par Hung Nguyen Viet.*

Soient $x_1, x_2, \ldots, x_n$ des nombres réels tels que

$$\sin x_1 \cos x_2 + \sin x_2 \cos x_3 + \cdots + \sin x_n \cos x_1 = \frac{n}{2}.$$

Démontrer que

$$\cos 2x_1 + \cos 2x_2 + \cdots + \cos 2x_n = 0.$$

4270. *Proposé par Leonard Giugiuc.*

Soient k et t des nombres réels tels que $k \in (0,1)$ et $t \in [\frac{\pi}{4}, \frac{\pi}{2}]$. Démontrer que

$$\int_0^t \frac{\cos x}{x^k} dx \geq \int_0^t \frac{\sin x}{x^k} dx.$$

Math Quotes

I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere. But I discovered that many mathematical demonstrations, which my teachers expected me to accept, were full of fallacies, and that, if certainty were indeed discoverable in mathematics, it would be in a new field of mathematics, with more solid foundations than those that had hitherto been thought secure. But as the work proceeded, I was continually reminded of the fable about the elephant and the tortoise. having constructed an elephant upon which the mathematical world could rest, I found the elephant tottering, and proceeded to construct a tortoise to keep the elephant from falling. But the tortoise was no more secure than the elephant, and after some twenty years of very arduous toil, I came to the conclusion that there was nothing more that I could do in the way of making mathematical knowledge indubitable.

Bertrand Russell in "Portraits from Memory."

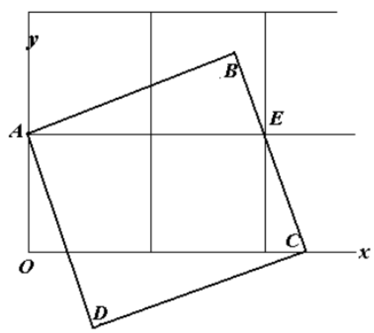
SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2016: 42(7), p. 313–317.

4161. *Proposed by Peter Y. Woo.*

A high-school math teacher discovered some geometry problems while sliding a rug under his feet, over a floor with square tiles of length 1 unit. He chose x and y axes along two edges of some arbitrary tile. Today, he moved the square rug $ABCD$ of length between 1 and 2 units, so that A is on $(0,1)$ and C is on $(c,0)$ for some $c > 2$. Then surprise! He noticed that the edge BC goes through the point $(2,1)$. Can you find $\angle BAE$?



We received 34 *submissions, out of which* 33 *were complete and correct. We present two of the most elegant solutions.*

Solution 1, by Kee-Wai Lau, slightly extended by the editor.

Let F be the point $(2,0)$ and let $\theta = \angle BAE = \angle FEC$. Then $AB = 2\cos\theta$, $BE = 2\sin\theta$ and

$$EC = BC - BE = AB - BE = 2(\cos\theta - \sin\theta).$$

Thus

$$1 = EF = EC\cos\theta = 2\cos\theta(\cos\theta - \sin\theta),$$

or

$$\cos(2\theta) - \sin(2\theta) = 0,$$

which implies

$$\tan(2\theta) = 1$$

and hence $\theta = \frac{\pi}{8}$.

Solution 2, by Ivko Dimitrić, slightly modified by the editor.

Let the vertical line $x = 1$ intersect the lines AE and AB at points P and Q respectively, and let F be the point $(2, 0)$. Then the right triangles APQ and EFC are congruent, having two congruent angles $\angle QAP = \angle CEF$ and adjacent legs each of unit length. Hence $AQ = EC$, and thus $BQ = BE$, which implies that QEB is a right isosceles triangle. Therefore

$$\angle AQE = \pi - \angle BQE = \frac{3\pi}{4}.$$

Since PQ halves AE at a right angle, the triangle AQE is also isosceles, and hence

$$\angle BAE = \angle QAE = \frac{\pi}{8}.$$

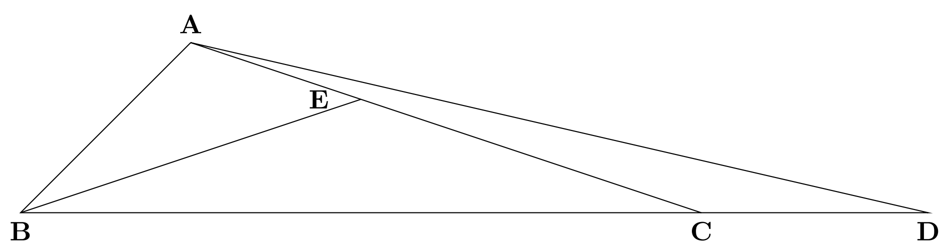
4162. *Proposed by George Apostolopoulos.*

Let ABC be a triangle such that $\angle B = 2\angle C$. We extend the side BC by a segment CD equal to $\frac{1}{3}BC$. Prove that

$$\text{Area}(ABC) = \frac{1}{4}|BC|^2 \cdot \cot\frac{\theta}{2},$$

where $\theta = \angle BAD$.

There were 19 correct solutions. We present five different solutions. The first two solutions are adapted from those of Daniel Dan, George Florin Serban, Titu Zvonaru and the proposer.



Solution 1.

Let a, b, c, d be the respective lengths of BC, CA, AB, AD, and suppose that the bisector of angle ABC intersects AC at E. Then, from the similarity of triangles AEB and ABC and the equality $BE = EC = ab(a + c)^{-1}$, we have that $AC : AB = BC : BE$, whence

$$b^2 = c(a + c). \tag{1}$$

Applying Stewart's Theorem to triangle ABD and cevian AC (or alternatively the Cosine Law to triangles ABC and ACD and eliminating the cosine of the angle at C), we find that

$$ad^2 + \left(\frac{a}{3}\right)c^2 = \left(b^2 + \frac{a^2}{3}\right)\left(\frac{4a}{3}\right).$$

This, along with (1), leads to $9d^2 = 4a^2 + 12b^2 - 3c^2 = (3c + 2a)^2$, whence

$$d = \frac{1}{3}(3c + 2a). \tag{2}$$

Applying the Cosine Law to triangle ABD and using (2) yields that

$$16a^2 = 9(c^2 + d^2 - 2cd\cos\theta) = (18c^2 + 12ac + 4a^2) - 6c(3c + 2a)\cos\theta,$$

so that

$$6(3c^2 + 2ac)\cos\theta = 6(3c^2 + 2ac - 2a^2),$$

whence

$$\cos\theta = 1 - \frac{2a^2}{3c^2 + 2ac} = 1 - \frac{2a^2}{3cd}, \tag{3}$$

and

$$cd = \frac{a^2}{3\sin^2\frac{\theta}{2}}.$$

Therefore

$$[ABC] = \frac{3}{4}[ABD] = \frac{3}{8}cd\sin\theta = \frac{1}{4}a^2\cot^2\frac{\theta}{2}.$$

Solution 2.

We begin by deriving an expression for $\cot\theta/2$. As in Solution 1, we can use (1) and (2) to derive (3), from which

$$1 - \cos\theta = \frac{2a^2}{3c^2 + 2ac}, \qquad 1 + \cos\theta = \frac{6c^2 + 4ac - 2a^2}{3c^3 + 2ac},$$

and

$$\cot^2\frac{\theta}{2} = \frac{1 + \cos\theta}{1 - \cos\theta} = \frac{(c + a)(3c - a)}{a^2}. \tag{4}$$

From Heron's area formula, (1) and (4), we obtain that

$$\begin{aligned}16[ABC]^2 &= (a + c + b)(a + c - b)(b - a + c)(b + a - c)\\ &= [(a + c)^2 - b^2][b^2 - (c - a)^2] = [a(a + c)][a(3c - a)]\\ &= a^4\cot^2\frac{\theta}{2}.\end{aligned}$$

whence $[ABC] = \frac{a^2}{4}\cot\frac{\theta}{2}$ as desired.

Solution 3, by C.R. Pranesachar.

Using the notation of Solution 1 and the fact that $3d = 2a + 3c$, we note that the triangle ABD has sides c, d and $4a/3$ and semiperimeter $a + c$. Applying the formula

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{(s-b)(s-c)}{[ABC]}$$

for an arbitrary triangle ABC to the triangle ABD and angle θ, we obtain

$$\cot\frac{\theta}{2} = \frac{[ABD]}{a^2/3} = \frac{4[ABC]}{a^2}$$

as desired.

Solution 4, by Leonard Giugiuc.

Place the triangle in the Cartesian plane with A at $(0,1)$ and C at $(k,0)$ with $k = \cot C > 0$. We find that the coordinates of A, B, C, D are given by

$$A(0,1), \quad B\left(\frac{1-k^2}{2k}, 0\right), \quad C(k,0), \quad D\left(\frac{9k^2-1}{6k}, 0\right),$$

and that $a = (3k^2-1)/(2k) = 2[ABC]$. Since $x = \cot\frac{1}{2}\angle BDA$ is the positive solution of the equation

$$\frac{x^2-1}{2x} = \frac{9k^2-1}{6k},$$

we have $x = 3k$ and $\tan\frac{1}{2}\angle BDA = 1/(3k)$. Observe that $\theta + B + \angle BDA = \pi$ so that

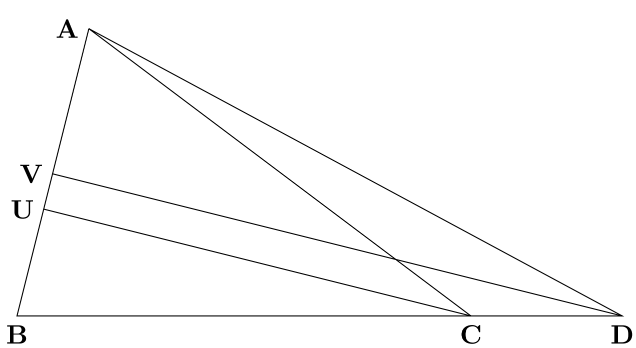
$$\cot\frac{\theta}{2} = \tan\left(C + \frac{\angle BDA}{2}\right) = \frac{4k}{3k^2-1} = \frac{2}{a}.$$

Therefore

$$\frac{1}{4}a^2\cot\frac{\theta}{2} = \frac{a}{2} = [ABC].$$

Solution 5, by Andrea Fanchini.

Let U and V be the respective feet of the perpendiculars from C and D to the side AB. Since $UV : BU = CD : BC = 1 : 3$, then $3(AU - AV) = 3UV = BU$.



(To cater to the various configurations, the lengths along the vector $\overrightarrow{AB}$ are signed, a contingency that can be accommodated using barycentric coordinates.) Since $DV : CU = BD : BC = 4 : 3$, then

$$\cot\theta = \frac{AV}{DV} = \frac{3AV}{4CU} = \frac{1}{4}\left(\frac{3AU - BU}{CU}\right) = \frac{1}{4}(3\cot A - \cot B).$$

Let $S = 2[ABC]$, twice the area of the triangle, and define $S_\phi = S\cot\phi$ for any angle ϕ. In particular

$$S_A = \frac{S\cos A}{\sin A} = bc\cos A = \frac{1}{2}(b^2 + c^2 - a^2),$$

with analogous expressions for S_B and S_C. It follows that

$$S_B + S_C = a^2 \quad \text{and} \quad 4S_\theta = 3S_A - S_B.$$

Since

$$2\cot B\cot C = 2\cot 2C\cot C = \cot^2 C - 1,$$

we have

$$2S_BS_C = S_C^2 - S^2,$$

so that

$$\begin{aligned} a^2S_B = S_B(S_B + S_C) &= (S_B + S_C)^2 + (2S_BS_C - S_C^2) - 3S_BS_C \\ &= a^4 - S^2 - 3S_BS_C = a^4 + 3(S^2 - S_BS_c) - 4S^2. \end{aligned}$$

From the identity $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$, we obtain

$$a^2S_B = a^4 + 3S_A(S_B + S_C) - 4S^2 = a^4 + 3a^2S_A - 4S^2,$$

which can be rearranged to yield

$$\frac{3S_A - S_B}{4} = \frac{4S^2 - a^4}{4a^2}.$$

We turn to the desired equality, which can be recast as

$$S_{\frac{\theta}{2}} = \frac{2S^2}{a^2}.$$

Since, for $0 < \phi < \pi$, there is a one-one relation between $S_\phi = (S_{\phi/2}^2 - S^2)/(2S_\phi/2)$ and $S_{\phi/2}$, the equality is equivalent to

$$\frac{3S_A - S_B}{4} = S_\theta = \frac{(2S^2/a^2)^2 - S^2}{4S^2/a^2} = \frac{4S^2 - a^4}{4a^2},$$

which has already been established.

Editor's Comments. The solutions revealed several interesting relations, in particular expressions for the cotangent in terms of the elements of triangle ABD:

$$\cot\frac{\theta}{2} = \frac{2\sin C\sin 2c}{\sin 3C} = \frac{\sqrt{2b^2+c^2-a^2}}{a}.$$

Václav Konečný pointed out that if angle C is less than 45°, then drawing the circle with centre A through B along with the radius to its other intersection with BC gives the diagram for the trisection of the exterior angle at A with a marked straightedge and compasses.

Miguel Amengual Covas observed that the condition $\angle B = 2\angle C$ figured in earlier problems published in ***Crux*** and references to those can be found in the article *Recurring* ***Crux*** *Configurations: No. 7* by Chris Fisher in Volume 38(6) of June 2012, p. 238–240.

4163. *Proposed by Leonard Giugiuc.*

Let a, b be real numbers with $0 < a < b$ and consider a positive sequence x_n such that

$$\lim_{n\to\infty}\left(ax_n + \frac{b}{x_n}\right) = 2\sqrt{ab}.$$

Find $\lim_{n\to\infty} x_n$ or show that it does not exist.

We received 17 submissions, all of which were correct. We present two different solutions.

Solution 1, by Adnan Ali.

Define the sequence $\{y_n\}$ as $y_n = ax_n + \dfrac{b}{x_n}$. Then $\lim_{n\to\infty} y_n = 2\sqrt{ab}$. Solving

$$ax_n^2 - y_n x_n + b = 0,$$

we have

$$x_n = \frac{1}{2a}\left(y_n \pm \sqrt{y_n^2 - 4ab}\right).$$

Since

$$\lim_{n\to\infty}\sqrt{y_n^2 - 4ab} = 0,$$

it follows that

$$\lim_{n\to\infty} x_n = \frac{1}{2a}\lim_{n\to\infty} y_n = \frac{2\sqrt{ab}}{2a} = \sqrt{\frac{b}{a}}.$$

Solution 2, by Michel Bataille.

From

$$\left(ax_n + \frac{b}{x_n}\right)^2 - \left(ax_n - \frac{b}{x_n}\right)^2 = 4ab$$

and the hypothesis, we deduce that

$$\lim_{n\to\infty}\left(ax_n - \frac{b}{x_n}\right)^2 = 0, \quad \text{so} \quad \lim_{n\to\infty}\left(ax_n - \frac{b}{x_n}\right) = 0.$$

Since

$$2ax_n = \left(ax_n + \frac{b}{x_n}\right) + \left(ax_n - \frac{b}{x_n}\right),$$

it follows that $\lim_{n\to\infty} 2ax_n = 2\sqrt{ab}$, and so $\lim_{n\to\infty} x_n = \sqrt{\dfrac{b}{a}}$.

Editor's comment. Both Roy Barbara and Oliver Geupel pointed out that the assumption $a < b$ is superfluous.

4164. *Proposed by G. Di Bona, A. Fiorentino, A. Moscariello and G. G. N. Angilella.*

In an election, N voters are to elect k representatives. Each voter must indicate exactly m distinct preferences, with $m \le k < N$. Every voter is a candidate themselves, and all candidates have a distinct age. The candidates are then ranked according to the number of votes received, and the k candidates who receive the largest number of votes are elected. In case of degeneracies, the eldest candidate is elected.

What is the minimum number of votes that a candidate should receive, in order to be sure to get elected?

There were five submissions, but only that of the proposer was complete and correct. The faulty solutions tended to argue from extreme situations without making a solid connection to the general one, however intuitively appealing this might be. Two solvers gave answers that depended on the status of particular individuals, rather than a value V that applied to any of the candidates. We present the solution of the proposer.

The minimum number of votes that a candidate should receive in order to ensure election is

$$V = \left\lfloor \frac{Nm}{k+1} \right\rfloor + 1.$$

First, we show that a candidate receiving at least V votes will be elected. Let $v_1, v_2, \ldots, v_{k+1}$ be the number of votes received by the top $k+1$ candidates, with $v_1 \ge v_2 \ge \cdots \ge v_{k+1}$. Since $v_1 + v_2 + \cdots + v_{k+1} \le Nm$, the total number of votes cast, the arithmetic mean M of these numbers must satisfy

$$v_{k+1} \le M \le \frac{Nm}{k+1} < V.$$

Hence, any candidate with at least V votes must be among the top k candidates and so be elected.

On the other hand, we construct a situation in which a candidate with $V-1$ votes fails to get elected. Suppose only $k+1$ candidates receive votes, and we number them $1, 2, \ldots, k+1$. Create a sequence with Nm terms by repeating the base sequence $\{1, 2, \ldots, k+1\}$ as many times as needed to fill the Nm slots. Partition this sequence into N subsequences of m terms, and let the ith voter vote for the candidates in the ith subsequence. Each candidate will receive at least $\lfloor \frac{Nm}{k+1} \rfloor = V-1$ votes and the candidate numbered $k+1$ will receive exactly $V-1$ votes. Thus, this candidate may fail to get elected.

Note that the role of the age rule is to break a tie. Consider the situation where the N voters are arranged in a circle and each votes for the m candidates sitting immediately to the right. Then each candidate receives the same number m of votes, and we need to decide on the winners by seniority.

4165. *Proposed by Daniel Sitaru.*

Prove that for all real numbers x_1, x_2, x_3 and x_4, we have,

$$|x_1+x_2+x_3+x_4| + 2(|x_1|+|x_2|+|x_3|+|x_4|) \geq 6\sqrt[6]{\prod_{1\leq i<j\leq 4} |x_i+x_j|}.$$

We received six submissions, all of which were correct. We present the solution with generalization by Michel Bataille.

We prove the stronger result that for any complex numbers x_1, x_2, x_3 and x_4, we have

$$|x_1+x_2+x_3+x_4| + 2(|x_1|+|x_2|+|x_3|+|x_4|) \geq \sum_{1\leq i<j\leq 4} |x_i+x_j|. \qquad (1)$$

The proposed inequality then follows from (1) by the AM-GM Inequality. To prove (1), we will make use of Hlawka's inequality which states that

$$|a+b+c| + |a| + |b| + |c| \geq |a+b| + |b+c| + |c+a| \qquad (2)$$

for all complex numbers a, b, c. (See e.g., problem **2482** [1999 : 430 ; 2000 : 506].) Setting $a = x_1, b = x_2$ and $c = x_3+x_4$, then from (2) we have

$$|x_1+x_2+x_3+x_4|+|x_1|+|x_2|+|x_3+x_4| \geq |x_1+x_2|+|x_2+x_3+x_4|+|x_1+x_3+x_4|. \qquad (3)$$

Applying (2) again, we obtain

$$|x_2+x_3+x_4| \geq |x_2+x_3| + |x_3+x_4| + |x_2+x_4| - |x_2| - |x_3| - |x_4| \qquad (4)$$

and

$$|x_1+x_3+x_4| \geq |x_1+x_3| + |x_3+x_4| + |x_1+x_4| - |x_1| - |x_3| - |x_4| \qquad (5)$$

Adding (4) and (5) and denoting the right side of (3) by R, then we have

$$R \geq |x_3+x_4| - |x_1| - |x_2| - 2|x_3| - 2|x_4| + \sum_{1\leq i<j\leq 4} |x_i+x_j|. \qquad (6)$$

From (3) and (6), we deduce that

$$\begin{aligned}&|x_1+x_2+x_3+x_4|+|x_1|+|x_2|+|x_3+x_4|\\&\quad\geq |x_3+x_4|-|x_1|-|x_2|-2|x_3|-2|x_4|+\sum_{1\leq i<j\leq 4}|x_i+x_j|\end{aligned}$$

from which (1) follows immediately.

4166. *Proposed by Mihaela Berindeanu.*

Show that for all real numbers x, y and z, we have:

$$2^{4x-y}+2^{4y-z}+2^{4z-x}\geq 2^{x+2y}+2^{y+2z}+2^{z+2x}.$$

We received 14 solutions. We present three solutions.

Solution 1, by Adnan Ali.

Let $2^x=a, 2^y=b$ and $2^z=c$. Then the proposed inequality becomes

$$\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{a}\geq ab^2+bc^2+ca^2,$$

which is nothing but a consequence of the Cauchy-Schwarz Inequality:

$$\left(\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{a}\right)(bc^2+ca^2+ab^2)\geq (a^2c+b^2a+c^2b)^2=(bc^2+ca^2+ab^2)^2.$$

Equality holds if and only if $a=b=c$ or equivalently $x=y=z$.

Solution 2, by Salem Malikić.

Introducing substitution $2x=a$, $2y=b$ and $2z=c$ our inequality becomes equivalent to

$$\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{a}\geq ab^2+bc^2+ca^2.$$

The last one follows after adding up the following obvious inequalities

$$\frac{a^4}{b}+bc^2\geq 2a^2c,\quad \frac{b^4}{c}+ca^2\geq 2b^2a,\quad \frac{c^4}{a}+ab^2\geq 2c^2b.$$

In order to achieve equality one must have

$$\frac{a^4}{b}=bc^2,\quad \frac{b^4}{c}=ca^2,\quad \frac{c^4}{a}=ab^2.$$

That implies $a^2=bc$, $b^2=ca$ and $c^2=ab$, i.e.

$$(a-b)^2+(b-c)^2+(c-a)^2=0,$$

so $a=b=c$, i.e. $x=y=z$.

Solution 3, by Michel Bataille.

Multiplying both sides by the positive real number $2^{-(x+y+z)}$, we obtain

$$2^{3x-2y-z} + 2^{3y-2z-x} + 2^{3z-2x-y} \geq 2^{y-z} + 2^{z-x} + 2^{x-y}$$

or, setting $a = 2^{y-z},\ b = 2^{z-x},\ c = 2^{x-y}$,

$$ac^3 + ba^3 + cb^3 \geq a + b + c. \tag{1}$$

Thus, it suffices to prove (1) for positive reals a, b, c such that $abc = 1$. Since $ac^3 = c(abc)\frac{c}{b} = c \cdot \frac{c}{b}$ and similarly $ba^3 = a \cdot \frac{a}{c},\ cb^3 = b \cdot \frac{b}{a}$, (1) rewrites as $X \geq 1$ where

$$X = \frac{a}{a+b+c} \cdot f\left(\frac{c}{a}\right) + \frac{b}{a+b+c} \cdot f\left(\frac{a}{b}\right) + \frac{c}{a+b+c} \cdot f\left(\frac{b}{c}\right)$$

where the function f is defined by $f(t) = \frac{1}{t}$.

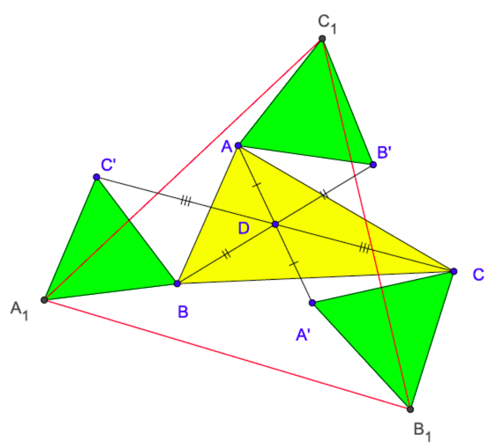
Now, since f is convex on the interval $(0, \infty)$, Jensen's inequality yields

$$X \geq f\left(\frac{a}{a+b+c} \cdot \frac{c}{a} + \frac{b}{a+b+c} \cdot \frac{a}{b} + \frac{c}{a+b+c} \cdot \frac{b}{c}\right) = f(1) = 1$$

and therefore (1) holds.

4167. *Proposed by Dao Thanh Oai and Leonard Giugiuc.*

Consider triangle ABC and let D be any point in the plane. Let points A', B', C' be reflections of points A, B, C in D, respectively. Construct the 3 triangles $AB'C_1$, $CA'B_1$ and $BC'A_1$ outwardly as the given diagram indicates:



Show that $A_1B_1C_1$ is an equilateral triangle.

We received 13 submissions, all correct. We feature two solutions; the first is typical of the ten that used complex coordinates.

Solution 1, by Somasundaram Muralidharan.

We must assume that the triangles $AB'C_1, CA'B_1$, and $AB'C_1$ are equilateral with the same orientation as ΔABC, but there is no loss of generality in assuming that D is at the origin. Let a, b, c be the complex numbers representing the vertices A, B, C respectively. Then A', B', C' are represented by the complex numbers $-a, -b, -c$ respectively. The vertex B_1 is obtained by rotating the segment CA' about vertex C anticlockwise through $60°$. Hence, if the complex number representing B_1 is b_1, then

$$b_1 = c + ((-a) - c)e^{\frac{i\pi}{3}} = c - (a + c)e^{\frac{i\pi}{3}}.$$

Similarly the complex numbers representing C_1 and A_1, namely c_1 and a_1, are given by

$$c_1 = a - (a + b)e^{\frac{i\pi}{3}}, \qquad a_1 = b - (b + c)e^{\frac{i\pi}{3}}.$$

Now,

$$\begin{aligned} \overrightarrow{C_1A_1} &= (b - (b + c)e^{\frac{i\pi}{3}}) - (a - (a + b)e^{\frac{i\pi}{3}}) \\ &= (b - a) + (a - c)e^{\frac{i\pi}{3}} \\ &= b - ce^{\frac{i\pi}{3}} - a(1 - e^{\frac{i\pi}{3}}) \\ &= b - ce^{\frac{i\pi}{3}} - ae^{-\frac{i\pi}{3}} \end{aligned}$$

Similarly,

$$\overrightarrow{C_1B_1} = -a + be^{\frac{i\pi}{3}} + ce^{-\frac{i\pi}{3}}.$$

Because

$$e^{\frac{i2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = -\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -e^{-\frac{i\pi}{3}},$$

we have

$$\overrightarrow{C_1B_1} = \overrightarrow{C_1A_1}e^{\frac{i\pi}{3}}.$$

Thus $\overrightarrow{C_1B_1}$ is obtained by rotating $\overrightarrow{C_1A_1}$ anticlockwise by $60°$ and the triangle $A_1B_1C_1$ is equilateral.

Solution 2, by Peter Y. Woo, with small corrections supplied by the editor.

Because the hexagon $AC'BA'CB'$ is symmetric about D, the quadrilaterals $BC'B'C$ and $ABA'B'$ are parallelograms. Compare the vector $\overrightarrow{B_1A_1}$, which is the sum of vectors $\overrightarrow{B_1A'}, \overrightarrow{A'B}$, and $\overrightarrow{BA_1}$, to the vector $\overrightarrow{B_1C_1}$, which is the sum of vectors $\overrightarrow{B_1C}, \overrightarrow{CB'}$, and $\overrightarrow{B'C_1}$:

$\overrightarrow{B_1A'}$ equals $\overrightarrow{B_1C}$ rotated $60°$ anticlockwise,
$\overrightarrow{A'B}$ equals $\overrightarrow{B'C_1}$ rotated $60°$ anticlockwise,
$\overrightarrow{BA_1}$ equals $\overrightarrow{CB'}$ rotated $60°$ anticlockwise.

Because vector addition is commutative, the sum of the first vector of each of the three pairs equals the sum of the last three vectors rotated 60° anticlockwise. Consequently $\overrightarrow{A_1B_1}$ is the vector $\overrightarrow{C_1B_1}$ rotated 60° anticlockwise. It follows that $A_1B_1C_1$ is an equilateral triangle.

4168. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 18$ and $abc = 4$. Prove that

$$6 \leq a + b + c \leq 2\sqrt{2\sqrt{6} + 4} + \sqrt{6} - 2.$$

When does equality hold?

We received 15 solutions, all of which were correct. We present the one by Joseph DiMuro.

We'll use Lagrange multipliers to find the extreme values of $f(a, b, c) = a + b + c$, subject to the constraints $g_1(a, b, c) = a^2 + b^2 + c^2 = 18$ and $g_2(a, b, c) = abc = 4$.

Note first that the extreme values must occur at points where $\bigtriangledown f = \lambda_1 \bigtriangledown g_1 + \lambda_2 \bigtriangledown g_2$ for some real numbers λ_1, λ_2, unless they occur at points where the vectors $\bigtriangledown g_1$ and $\bigtriangledown g_2$ are linearly dependent.

Since $\bigtriangledown f = (1, 1, 1,])$, $\bigtriangledown g_1 = (2a, 2b, 2c)$, and $\bigtriangledown g_2 = (bc, ca, ab)$, $\bigtriangledown g_1$ and $\bigtriangledown g_2$ are linearly dependent only if $a = b = c$. But the constraints $a^2 + b^2 + c^2 = 18$ and $abc = 4$ cannot be satisfied if $a = b = c$ since the system $3a^2 = 18$ and $a^3 = 4$ has no solutions.

Hence it suffices to look for points that satisfy $\bigtriangledown f = \lambda_1 \bigtriangledown g_1 + \lambda_2 \bigtriangledown g_2$. That is, $\lambda_1(2a, 2b, 2c) + \lambda_2(bc, ca, ab) = (1, 1, 1)$, or

$$2a\lambda_1 + bc\lambda_2 = 1 \quad (1)$$
$$2b\lambda_1 + ac\lambda_2 = 1 \quad (2)$$
$$2c\lambda_1 + ab\lambda_2 = 1 \quad (3)$$

From (1)-(2) we obtain

$$2(a - b)\lambda_1 + c(b - a)\lambda_2 = 0 \quad \iff \quad 2(a - b)\lambda_1 = c(a - b)\lambda_2,$$

which is true if and only if either $a = b$ or $2\lambda_1 = c\lambda_2$. Similarly, from (2)-(3) and (3)-(1) we deduce that either $b = c$ or $2\lambda_1 = a\lambda_2$; and either $c = a$ or $2\lambda_1 = b\lambda_2$.

Recall that $a = b = c$ is impossible. On the other hand, if a, b, and c are all distinct, then $2\lambda_1 = c\lambda_2 = b\lambda_2 = a\lambda_2$ would yield $\lambda_1 = \lambda_2 = 0$ so $\bigtriangledown f = 0$, a contradiction. Hence, exactly two of a, b, and c must be equal.

Without loss of generality, assume $a \neq b = c$. Then the original constraints become

$$a^2 + 2b^2 = 18 \quad \text{and} \quad ab^2 = 4,$$

which yields

$$\begin{aligned} &a^2 + \frac{8}{a} = 18, \\ &a^3 - 18a + 8 = 0, \\ &(a-4)(a^2+4a-2) = 0. \end{aligned}$$

Hence, $a = 4$ or $a = -2 \pm \sqrt{6}$. Since $a > 0$, we obtain $a = 4$ or $\sqrt{6} - 2$.

If $a = 4$, then $b = c = 1$, and if $a = \sqrt{6} - 2$, then $b = c = \sqrt{2\sqrt{6}+4}$. These points then yield the extreme values of

$$f(4,1,1) = 6 \quad \text{and} \quad f(\sqrt{6}-2, \sqrt{2\sqrt{6}+4}, \sqrt{2\sqrt{6}+4}) = 2\sqrt{2\sqrt{6}+4} + \sqrt{6} - 2.$$

Since

$$2\sqrt{2\sqrt{6}+4} + \sqrt{6} - 2 \approx 6.4157 > 6,$$

the results follow. Note that the maximum is attained at the 3 points obtained by permuting the coordinates of $(4,1,1)$ and the minimum is attained at the 3 points obtained by permuting the coordinates of

$$(\sqrt{6}-2, \sqrt{2\sqrt{6}+4}, \sqrt{2\sqrt{6}+4}).$$

4169. *Proposed by Michel Bataille.*

Let a, b, c be positive real numbers. Prove that

$$\left(a\sqrt{\frac{b}{a+b}} + b\sqrt{\frac{c}{b+c}} + c\sqrt{\frac{a}{c+a}}\right)\left(b\sqrt{\frac{a+b}{b}} + c\sqrt{\frac{b+c}{c}} + a\sqrt{\frac{c+a}{a}}\right) \le (a+b+c)^2.$$

There were 14 correct solutions submitted. Eight of these solvers independently gave the solution presented here.

Using the Cauchy-Schwarz and arithmetic-harmonic means inequalities, we obtain the following two inequalities:

$$\begin{aligned} a\sqrt{\frac{b}{a+b}} + b\sqrt{\frac{c}{b+c}} + c\sqrt{\frac{a}{c+a}} &= \sqrt{a}\sqrt{\frac{ab}{a+b}} + \sqrt{b}\sqrt{\frac{bc}{b+c}} + \sqrt{c}\sqrt{\frac{ca}{c+a}} \\ &= \sqrt{a+b+c}\sqrt{\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}} \\ &\le \sqrt{a+b+c}\sqrt{\frac{a+b}{4} + \frac{b+c}{4} + \frac{c+a}{4}} \\ &= \frac{1}{\sqrt{2}}(a+b+c) \end{aligned}$$

and

$$\begin{aligned} b\sqrt{\frac{a+b}{b}}+c\sqrt{\frac{b+c}{c}}+a\sqrt{\frac{c+a}{a}} &= \sqrt{b}\sqrt{a+b}+\sqrt{c}\sqrt{c+b}+\sqrt{a}\sqrt{c+a} \\ &\leq \sqrt{a+b+c}\sqrt{(a+b)+(b+c)+(c+a)} \\ &= \sqrt{2}(a+b+c). \end{aligned}$$

Multiplying these inequalities gives the desired result.

Editor's Comment. Nguyen Ngoc Tú used the concavity of the function $\sqrt{x}$ and reduced the problem to establishing that

$$ab(a+b)^{-1}+bc(b+c)^{-1}+ca(c+a)^{-1} \leq (a+b+c)/2.$$

4170. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let $ABCD$ be a circumscribed quadrilateral (that is, a quadrilateral for which an incircle can be constructed) and let P be the intersection point of AC and BD. Let h_a, h_b, h_c and h_d denote the distances from P to AB, BC, CD and DA, respectively. Prove that

$$\frac{AB\cdot CD}{AD\cdot BC}=\frac{h_b+h_d}{h_a+h_c}.$$

We received nine submissions, all correct, and will feature two of them. The first is typical of those that reduced the problem to a fairly recent result; the second is provided for those readers who prefer to see the details.

Solution 1, by Adnan Ali.

Let θ be the angle between the diagonals AC and BD. Then observe that we have the following relations:

$$\begin{aligned} AB\cdot h_a &= PA\cdot PB\sin\theta, \\ BC\cdot h_b &= PB\cdot PC\sin\theta, \\ CD\cdot h_c &= PC\cdot PD\sin\theta, \\ DA\cdot h_d &= PD\cdot PA\sin\theta. \end{aligned}$$

It follows that

$$\frac{AB\cdot CD}{AD\cdot BC}=\frac{h_b\cdot h_d}{h_a\cdot h_c}.$$

The desired result is thereby reduced to proving

$$\frac{h_b\cdot h_d}{h_a\cdot h_c}=\frac{h_b+h_d}{h_a+h_c}.$$

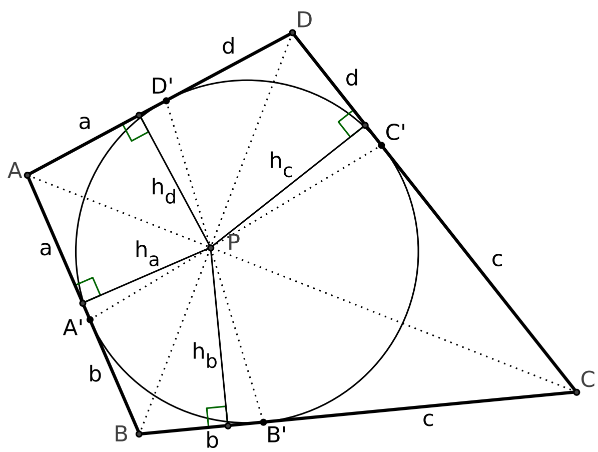
But this is an immediate consequence of a known theorem, namely, a convex quadrilateral $ABCD$ has an incircle if and only if the altitudes h_a, h_b, h_c, h_d from the intersection point of the diagonals to each of the four sides satisfy

$$\frac{1}{h_a}+\frac{1}{h_c}=\frac{1}{h_b}+\frac{1}{h_d}.$$

Proofs of the theorem can be found in [2], for example. Moreover, this result was Problem 5 on the 2015 Indian National Mathematics Olympiad.

Solution 2 is a composite of similar solutions from Oliver Geupel and John G. Heuver.

Let A', B', C', and D' be the points where the lines AB, BC, CD, and DA, respectively, touch the incircle Γ, and denote the tangent lengths by $a = AA' = AD'$, $b = BB' = BA'$, $c = CC' = CB'$, and $d = DD' = DA'$. A degenerate version of Brianchon's theorem tells us that the chords $A'C'$ and $B'D'$ also pass through P. Here is an alternative proof of this claim: Let Q be the intersection point of the lines AC and $A'C'$.



The tangents AB and CD to Γ are symmetric with respect to the perpendicular bisector of the chord $A'C'$. Hence $\angle QA'A = \angle DC'Q = 180^\circ - \angle QC'C$. By the Sine Law applied to triangles AQA' and CQC' we obtain

$$\frac{AQ}{AA'} = \frac{\sin\angle QA'A}{\sin\angle AQA'} = \frac{\sin\angle QC'C}{\sin\angle CQC'} = \frac{CQ}{CC'};$$

that is, $AQ : CQ = a : c$. Similarly, if R is the intersection point of AC and $B'D'$, we have $AR : CR = a : c$. It follows that $Q = R$ so that the lines AC, $A'C'$, and $B'D'$ are concurrent at point $Q = R$. Similarly, the lines BD, $A'C'$ and $B'D'$ are concurrent at $Q = R$. Thus, the lines AC, BD, $A'C'$, and $B'D'$ are concurrent at $P = Q = R$. We deduce, therefore, that

$$\frac{AP}{PC} = \frac{a}{c} \quad \text{and} \quad \frac{BP}{PD} = \frac{b}{d}.$$

It follows that $\frac{[ABP]}{[BCP]} = \frac{AP}{PC} = \frac{a}{c}$, where we have used square brackets for the area of a triangle. Analogous results involving the other relevant triangles give us

$$\frac{[ABP]}{ab} = \frac{[BCP]}{bc} = \frac{[CDP]}{cd} = \frac{[DAP]}{da}. \tag{1}$$

Note that $2[ABP] = (a+b)h_a$, so that

$$h_a = \frac{2[ABP]}{a+b} = \frac{2kab}{a+b}, \tag{2}$$

where we have set k equal to the common ratio in (1). Similar formulas hold for the remaining triangles. We finally obtain

$$\begin{aligned}\frac{AB \cdot CD}{AD \cdot BC} &= \frac{(a+b)(c+d)}{(d+a)(b+c)} \\ &= \frac{\frac{1}{(b+c)} \cdot \frac{1}{(d+a)}}{\frac{1}{(a+b)} \cdot \frac{1}{(c+d)}} \\ &= \frac{\frac{bc}{b+c} + \frac{da}{d+a}}{\frac{ab}{a+b} + \frac{cd}{c+d}} = \frac{h_b + h_d}{h_a + h_c},\end{aligned}$$

which completes the proof.

Editor's comments. Note that equation (2) (together with the corresponding equations for h_b, h_c, h_d) lead immediately to a proof of the theorem mentioned in the first solution. Other proofs of the theorem can be found in [1], where Josefsson traces the result back to a 1995 problem in the Russian Journal *Kvant* [3]. He reports that it also appeared in [4], and as a problem in the 4th stage of the 48th German Mathematical Olympiad (2009).

References

[1] Martin Josefsson, Similar Metric Characterizations of Tangential and Extangential Quadrilaterals, *Forum Geometricorum*, **12** (2012) 63–77.

[2] N. Minculete, Characterizations of a Tangential Quadrilateral, *Forum Geometricorum*, **9** (2009) 113–118.

[3] I. Vaynshtejn, N. Vasilyev and V. Senderov, Problem M1495, *Kvant* (in Russian) no. 6 (1995) 27–28.

[4] A. Zaslavsky, Problem M1887, *Kvant* (in Russian) no. 3 (2004) p. 19.

AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Solvers - individuals

Solvers - collaborations