THE CONTEST CORNER

No. 57

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The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **March 1, 2018**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC281. In the Original Six era of the NHL, one particular season, each team played 20 games (each team played the other 5 teams 4 times each). Each game ended as a win, a loss or a tie (there were no 'overtime losses'). At the end of this certain season, the standings were as below. What were all the possible outcomes for Montreal's number of wins X, losses Y and ties Z?

Team	Wins	Losses	Ties
Toronto	2	12	6
Boston	6	10	4
Detroit	7	12	1
New York	7	9	4
Chicago	11	7	2
Montreal	X	Y	Z

CC282. Calculate the value of

$$\left(3^{4/3}-3^{1/3}\right)^3+\left(3^{5/3}-3^{2/3}\right)^3+\left(3^{6/3}-3^{3/3}\right)^3+\cdots+\left(3^{2006/3}-3^{2003/3}\right)^3.$$

CC283. Two bags, Bag A and Bag B, each contain 9 balls. The 9 balls in each bag are numbered from 1 to 9. Suppose one ball is removed randomly from Bag A and another ball from Bag B. If X is the sum of the numbers on the balls left in Bag A and Y is the sum of the numbers of the balls remaining in Bag B, what is the probability that X and Y differ by a multiple of 4?

CC284. Define the function $f(x)$ to be the largest integer less than or equal to x for any real x. For example, $f(1)=1, f(3/2)=1, f(7/2)=3, f(7/3)=2$. Let

$$g(x)=f(x)+f(x/2)+f(x/3)+\cdots+f(x/(x-1))+f(x/x).$$

a) Calculate $g(4)-g(3)$ and $g(7)-g(6)$.

b) What is $g(116)-g(115)$?

CC285. Find all values of k so that $x^2 + y^2 = k^2$ will intersect the circle with equation $(x-5)^2 + (y+12)^2 = 49$ at exactly one point.

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CC281. À l'époque des six premières équipes de la LNH, lors d'une saison particulière, chaque équipe jouait 20 matchs (chaque équipe rencontrait chacune des 5 autres équipes 4 fois). Chaque match se terminait par une victoire, une défaite ou un match nul (il n'y avait aucun jeu en temps supplémentaire). Le tableau suivant présente le classement à la fin de cette saison. Quels sont tous les résultats possibles quant au nombre X de victoires, au nombre Y de défaites et au nombre Z de matchs nuls de l'équipe de Montréal?

Équipe	Victoires	Défaites	Matchs nuls
Toronto	2	12	6
Boston	6	10	4
Detroit	7	12	1
New York	7	9	4
Chicago	11	7	2
Montréal	X	Y	Z

CC282. Calculer la valeur de

$$\left(3^{4/3} - 3^{1/3}\right)^3 + \left(3^{5/3} - 3^{2/3}\right)^3 + \left(3^{6/3} - 3^{3/3}\right)^3 + \cdots + \left(3^{2006/3} - 3^{2003/3}\right)^3.$$

CC283. Deux sacs, A et B, contiennent chacun 9 boules. Dans chaque sac, les 9 boules sont numérotées de 1 à 9. On retire au hasard une boule du sac A et une boule du sac B. Soit X la somme des numéros sur les boules qui se trouvent encore dans le sac A et Y la somme des numéros sur les boules qui se trouvent encore dans le sac B. Quelle est la probabilité pour que la différence entre X et Y soit un multiple de 4?

CC284. On définit la fonction f sur l'ensemble des réels comme suit: $f(x)$ est le plus grand entier inférieur ou égal à x. Par exemple, $f(1) = 1$, $f(\frac{3}{2}) = 1$, $f(\frac{7}{2}) = 3$ et $f(\frac{7}{3}) = 2$. Soit

$$g(x) = f(x) + f(x/2) + f(x/3) + \cdots + f(x/(x-1)) + f(x/x).$$

a) Calculer $g(4) - g(3)$ et $g(7) - g(6)$.

b) Quelle est la valeur de $g(116) - g(115)$?

CC285. Déterminer toutes les valeurs de k pour que le cercle d'équation $x^2 + y^2 = k^2$ et le cercle d'équation $(x-5)^2 + (y+12)^2 = 49$ se coupent en exactement un point.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(7), p. 291–293.

CC231. If $x^2 + y^2 = 6xy$ with $y > x > 0$, find $\frac{x+y}{x-y}$.

Originally Question 6 of The Seventh W.J. Blundon Contest, 1990.

We received 19 submissions of which 15 were correct and complete. We present 2 solutions.

Solution 1, by Dan Daniel.

We have

$$x^2 + y^2 = 6xy \iff 2x^2 + 2y^2 - 4xy = x^2 + y^2 + 2xy \iff 2(x-y)^2 = (x+y)^2,$$

so

$$\left(\frac{x+y}{x-y}\right)^2 = 2.$$

Since $y > x$, then $\dfrac{x+y}{x-y} = -\sqrt{2}$.

Solution 2, by Titu Zvonaru.

Let $t = \dfrac{y}{x}$. From $x^2 + y^2 = 6xy$ we obtain

$$t^2 - 6t + 1 = 0.$$

Since $t > 1$, we get

$$\frac{x+y}{x-y} = \frac{1+t}{1-t} = \frac{1+3+\sqrt{8}}{1-3-\sqrt{8}} = -\frac{2+\sqrt{2}}{1+\sqrt{2}} = -\sqrt{2}.$$

Editor's Comments. All the wrong submissions reported as a result $\sqrt{2}$ instead of $-\sqrt{2}$. Someone did a mistake when copying the problem (wrote $x > y$ instead of $y > x$), someone forgot that $y > x$, so when you take the square root of $(x-y)^2$ you get a negative number. Konstantine Zelator also considered the general case

$$x^2 + y^2 = kxy, \quad k \text{ is a real number bigger than } 2,$$

and proved that

$$\frac{x+y}{x-y} = -\sqrt{\frac{k+2}{k-2}}.$$

CC232. Seven tests are given and on each test no ties are possible. Each person who is the top scorer on at least one of the tests or who is in the top six on at least four of these tests is given an award, but each person can receive at most one award. Find the maximum number of people who could be given awards if 100 students take these tests.

Originally Team Question 3 of the 1988 Florida Mathematics Olympiad.

We received four correct solutions. We present a combination of all four solutions.

The maximum number of people who could be given awards is 15. There are always 7 top scorers who get an award. The other awards are given to people who were in the top six in at least 4 tests. Altogether 35 people are ranked 2nd, 3rd, 4th, 5th, and 6th. The maximum number of people who could be given awards will be reached if there are as many people who are four times in the top 6 as possible.

$35 = 4 \cdot 8 + 3$, so the maximum number of people who could get awards by being four times in the top 6 is 8. Thus, the maximum number of students that can be given an award is $7 + 8 = 15$; and this works as long as there are at least 15 test-takers, whereas number 100 does not play any special role. A specific set of outcomes with 15 awards can be realized by

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7
1^{st}	A	B	C	D	E	F	G
2^{nd}	H	H	H	H	I	I	I
3^{rd}	I	J	J	J	J	K	K
4^{th}	K	K	L	L	L	L	M
5^{th}	M	M	M	N	N	N	N
6^{th}	O	O	O	O	X	Y	Z

CC233. Let P be a point in the interior of the rectangle $ABCD$. Suppose that $PA = a, PB = b$ and $PC = c$, find, in terms of a, b, c, the length of the line segment PD.

Originally Individual Question 12 (b) of the 1988 Florida Mathematics Olympiad.

We received 13 correct solutions. We present the solution by Titu Zvonaru.

Let P_1 and P_2 be the projections of P onto AB and AD, respectively. By the Pythagorean Theorem,

$$(PP_1)^2+(PP_2)^2 = a^2, (AB-PP_1)^2+(PP_2)^2 = b^2, (AD-PP_2)^2+(AB-PP_1)^2 = c^2.$$

It follows that

$$(PD)^2 = (AD - PP_2)^2 + (AB - PP_1)^2 = a^2 + c^2 - b^2,$$

so that

$$PD = \sqrt{a^2 + c^2 - b^2}.$$

CC234. Find B if

$$x = \frac{\log_{10} 16/3}{\log_{10} B}$$

is the solution to the exponential equation

$$2^{2x+4} + 3^{3x+2} = 4^{x+3}.$$

Originally Individual Question 10 of the 1988 Florida Mathematics Olympiad.

We received 14 correct solutions and one incorrect solution. We present the solution by Kathleen Lewis.

The given equation $2^{2x+4} + 3^{3x+2} = 4^{x+3}$ can be rewritten as

$$3^{3x+2} = 4^{x+3} - 4^{x+2} = 3 \cdot 4^{x+2}.$$

Thus

$$3 \cdot 3^{3x} = 16 \cdot 4^{x},$$

so $27^x/4^x = 16/3$. Then

$$x \log_{10}(27/4) = \log_{10}(16/3),$$

so $B = 27/4$.

CC235. Find the area of a regular octagon formed by cutting equal isosceles triangles from the corners of a square with sides of one unit.

Originally Question 6 of The Ninth W.J. Blundon Contest, 1992.

We received 13 correct solutions. We present the solution by Kathleen Lewis.

To end up with a regular octagon with sides of length x, each triangle that is cut off will have legs of length $x/\sqrt{2}$. The original square was a unit square, so

$$1 = x + 2 \cdot x/\sqrt{2} = x(1 + \sqrt{2}),$$

implying that

$$x = \frac{1}{1 + \sqrt{2}} = \sqrt{2} - 1.$$

Each triangle that was cut off has area $x^2/4$, so the total area removed is $x^2 = 3 - 2\sqrt{2}$. Therefore, the remaining area is

$$1 - (3 - 2\sqrt{2}) = 2\sqrt{2} - 2.$$